MATHEMATICAL MODELLING: THEORY AND APPLICATIONS

AN INTRODUCTION TO ACTUARIAL MATHEMATICS

A.K. Gupta and T. Varga

Springer-Science+Business Media, B.V.

An Introduction to Actuarial Mathematics

MATHEMATICAL MODELLING: Theory and Applications

VOLUME 14

This series is aimed at publishing work dealing with the definition, development and application of fundamental theory and methodology, computational and algorithmic implementations and comprehensive empirical studies in mathematical modelling. Work on new mathematics inspired by the construction of mathematical models, combining theory and experiment and furthering the understanding of the systems being modelled are particularly welcomed.

Manuscripts to be considered for publication lie within the following, non-exhaustive list of areas: mathematical modelling in engineering, industrial mathematics, control theory, operations research, decision theory, economic modelling, mathematical programmering, mathematical system theory, geophysical sciences, climate modelling, environmental processes, mathematical modelling in psychology, political science, sociology and behavioural sciences, mathematical biology, mathematical ecology, image processing, computer vision, artificial intelligence, fuzzy systems, and approximate reasoning, genetic algorithms, neural networks, expert systems, pattern recognition, clustering, chaos and fractals.

Original monographs, comprehensive surveys as well as edited collections will be considered for publication.

The titles published in this series are listed at the end of this volume.

An Introduction to Actuarial Mathematics

by

A.K. Gupta
Bowling Green State University,
Bowling Green, Ohio, U.S.A.

and

T. Varga
National Pension Insurance Fund,
Budapest, Hungary

SPRINGER-SCIENCE+BUSINESS MEDIA, B.V.

A C.I.P. Catalogue record for this book is available from the Library of Congress.

ISBN 978-90-481-5949-9 ISBN 978-94-017-0711-4 (eBook)
DOI 10.1007/978-94-017-0711-4

Printed on acid-free paper

To
Alka, Mita, and Nisha
AKG

To
Terézia and Julianna
TV

TABLE OF CONTENTS

PREFACE

This text has been written primarily as an introduction to the basics of the actuarial mathematics of life insurance. The subject matter is suitable for a one-semester or a one-year college course. Since it attempts to derive the results in a mathematically rigorous way, the concepts and techniques of one variable calculus and probability theory have been used throughout. A two semester course in calculus and a one semester course in probability theory at the undergraduate level are the usual prerequisites for the understanding of the material.

There are five chapters in this text. Chapter 1 focuses on some important concepts of financial mathematics. The concept of interests, essential to the understanding of the book, is discussed here very thoroughly. After the study of present values in general, annuities-certain are examined.

Chapter 2 is concerned with the mortality theory. The analytical study of mortality is followed by the introduction of mortality tables.

Chapter 3 discusses different types of life insurances in detail. First, we examine stochastic cash flows in general. Then we study pure endowments, whole life and term insurances, endowments, and life annuities.

Chapter 4 is devoted to premium calculations. It opens with a section on net premiums, followed by a section discussing office premiums.

Chapter 5 deals with reserves. In addition to presenting different reserving methods, the mortality profit is also studied here. Careful consideration is given to the problem of negative reserves as well.

The book contains many systematically solved examples showing the practical applications of the theory presented. Solving the problems at the end of each section is essential for understanding the material. Answers to odd-numbered problems are given at the end of the book.

The authors are indebted to Drs. D. K. Nagar, Jie Chen and Asoka Ramanayake with whom discussions and whose comments on the original manuscript, resulted in a vastly improved text. We are grateful to the Society of Actuaries, the Institute of Actuaries and the Faculty of Actuaries, acknowledged in the appropriate places, for permission to reproduce tables. Thanks are also due to Ms. Anneke Pot of Kluwer Academic Publishers for her cooperation and help. Her counsel on matters of form and style is sincerely appreciated. Finally, the authors are thankful to Cynthia Patterson for her efficient word processing and for her patience during the preparation of this manuscript.

Bowling Green, Ohio — A. K. Gupta
Budapest, Hungary — T. Varga
August, 2001

CHAPTER 1

FINANCIAL MATHEMATICS

1.1 COMPOUND INTEREST

There are many situations in every day life when we come across the concept of interest. For example, when someone makes a deposit in a bank account, the money will earn interest. Money can also be borrowed from a bank, which has to be repaid later on with interest. Many car buyers who cannot afford paying the price in cash will pay monthly installments whose sum is usually higher than the original price of the car. This is due to the interest which has been added to the price of the car.

In general, interest is the fee one pays for the right of using someone else's money. In the case of a bank deposit, the bank uses the money of the depositor, whereas in the case of a bank loan, the borrower uses the money of the bank. The situation is slightly complicated with a car purchase. What happens here is that the buyer of the car gets a loan from the dealer which can be used to pay the price of the car in full. This loan has to be repaid to the dealer in monthly installments.

Consider an interest earning bank account. Assume we make a deposit of $\$X$ on January 1, and leave the money there until the end of the year. On January 1 of the next year, we withdraw the money together with the interest. We find that our money has accumulated to $\$Y$. What is the rate of interest for the year? Let us denote the interest rate by "i". It is worth noting that the interest rate is often expressed as a percentage (e.g. 4%), or as a real number (e.g. 0.04).

Let us split $\$Y$ into two parts. One is the amount of the original deposit: $\$X$, also called the *capital content*, the rest is the *interest content*: $\$Xi$. We have the following equation:

$$Y = X + Xi$$

or equivalently

$$Y = X(1 + i).$$

EXAMPLE 1.1. A sum of $340 is deposited in a bank account on January 1, 1990. The account earns interest at a rate of 3% per annum. What is the accumulated value of the account on January 1, 1991? What are the capital and interest contents of the accumulation?

Solution: The accumulated value is $\$340(1 + 0.03) = \$350.20$. The capital content is $\$340$. The interest is $\$340 \times 0.03 = \$10.20$. The interest could also be obtained as $\$350.20 - \$340$.

We may ask what happens if $\$X$ is deposited on January 1, but the withdrawal takes place earlier than January 1 of the next year. How much money can we withdraw then? The capital content must be the same; that is, \$340, but the whole annual interest cannot be paid. We could argue that the interest should be proportional to the time between the deposit and the withdrawal. Thus, if the withdrawal takes place n days after the deposit, the interest paid for this time interval is $\$X\frac{in}{365}$. So the amount available at withdrawal is $\$X\left(1 + i\frac{n}{365}\right)$. Interest computed in this way is called *simple interest*. Later on, we will see that the interest for a fraction of a year can be defined in another way as well.

EXAMPLE 1.2. A sum of \$340 is deposited in a bank account on January 1, 1990. The account earns interest at a rate of 3% per annum. Using simple interest, answer the following questions.

a) What is the accumulated value of the account on February 1, 1990? What is the interest paid for the period between January 1 and February 1?

b) What is the accumulated value of the account on July 1, 1990? What is the interest for the period from January 1 to July 1?

Solution: a) The number of days from January 1 to February 1 is $n = 31$. The interest for this period is $\$340(0.03)\frac{31}{365} = \$0.8663$. Hence, the accumulated value on February 1 is $\$340 + \$0.8663 = \$340.87$.

b) Since the number of days from January 1 to July 1 is $n = 31 + 28 + 31 + 30 + 31 + 30 = 181$ and the interest for this period is $\$340(0.03)\frac{181}{365} = \$5.0581$, the accumulation on July 1 is $\$340 + \$5.0581 = \$345.06$.

In Example 1.1, we found that a capital of \$340 invested on January 1, 1990 grows up to \$350.20 by January 1, 1991. Furthermore, Example 1.2 showed that the same investment accumulates to \$345.06 by July 1, 1990. Now, assume we withdraw this \$345.06 on July 1 and deposit it immediately afterwards. Then the amount will accumulate to $\$345.06\left(1 + 0.03\frac{365 - 181}{365}\right) = \$350.28$ by January 1, 1991. That means, we earned an interest of \$10.28 over the year which is higher than the \$10.20 interest earned in Example 1.1. Therefore, we can make a better deal, if instead of leaving our money in the account for the whole year, we withdraw and redeposit it in the middle of the year.

We can go further. If we withdraw the money from the account every day of the year and deposit it immediately afterwards, we earn interest at a rate of $\frac{0.03}{365}$ each day. Hence, the original \$340 will accumulate to $\$340\left(1+\frac{0.03}{365}\right)^{365} = \$350.35$ in one year. So we can earn an interest of \$10.35 in this way. This interest is higher than the interests obtained in the previous transactions. Hence, if we wanted to maximize the gain on our investment, we would have to run to the bank every day. Therefore, we introduce another type of interest which makes it possible to avoid this problem.

First, let us introduce some notations. If we invest one unit, say \$1, at time t_1, its accumulation (or accumulated value) at time t_2 is denoted by $A(t_1,t_2)$. We will measure time in years.

We call $A(t_1,t_2)$ the *accumulation factor*. We can also write $A(t_1,t_2) = 1 + i_{eff}(t_1,t_2)$ where $i_{eff}(t_1,t_2)$ is called the *effective rate of interest* for the term from t_1 to t_2. The notation "i" without subscript will be reserved for the annual interest rate. The problem with simple interest rate is that $A(t_1,t_2)\,A(t_2,t_3) \neq A(t_1,t_3)$. We would like to have a function $A(t_1,t_2)$ satisfying

$$A(t_1,t_2)\,A(t_2,t_3) = A(t_1,t_3), \text{ for } t_1 < t_2 < t_3. \qquad (1)$$

This is called the *principle of consistency.* It can also be written as

$$(1 + i_{eff}(t_1,t_2))\,(1 + i_{eff}(t_2,t_3)) = 1 + i_{eff}(t_1,t_3), \text{ for } t_1 < t_2 < t_3.$$

If $A(t_1,t_2)$ satisfies the consistency condition, the interest is called *compound interest*. Applying the above relationship repeatedly, we can see that it is equivalent to

$$A(t_1,t_2)\,A(t_2,t_3)\,\dots\,A(t_{n-1},t_n) = A(t_1,t_n), \text{ for } t_1 < t_2 < \dots < t_{n-1} < t_n,$$

or

$$(1 + i_{eff}(t_1,t_2))\,(1 + i_{eff}(t_2,t_3))\,\dots\,(1 + i_{eff}(t_{n-1},t_n)) = 1 + i_{eff}(t_1,t_n), \\ \text{for } t_1 < t_2 < \dots < t_{n-1} < t_n.$$

One of the most often used functions satisfying (1) is

$$A(t_1,t_2) = (1 + i)^{t_2-t_1}. \qquad (2)$$

Remember that t_1 and t_2 are measured in years. As we have mentioned earlier, i is the annual rate of interest. This function satisfies the principle of consistency, since

$$\begin{aligned} A(t_1,t_2)\,A(t_2,t_3) &= (1+i)^{t_2-t_1}\,(1+i)^{t_3-t_2} \\ &= (1+i)^{t_3-t_1} \\ &= A(t_1,t_3). \end{aligned}$$

EXAMPLE 1.3. A sum of \$340 is deposited in a bank account on January 1, 1990. The interest follows (2), and the annual rate of interests is 3%. What is the accumulation on July 1, 1990? If we withdraw and redeposit the accumulated amount on July 1, 1990, what will it grow up to by January 1, 1991?

Solution: The number of days from January 1 to July 1 is 181. Therefore $t_2 - t_1 = \frac{181}{365}$, and using (2) we get

$$A(t_1,t_2) = (1.03)^{\frac{181}{365}} = 1.0147658$$

and hence \$340 accumulates to $\$340 \times 1.0147658 = \$345.02$ by July 1.

The number of days from July 1, 1990 to January 1, 1991 is 184. Therefore, \$345.02 accumulates to $\$345.02(1.03)^{\frac{184}{365}} = \$350.20$ by January 1, 1991.

If we compare this result with Example 1.1, we can see that the deposit made on January 1, 1990 accumulated to the same amount with or without redepositing the money in the middle of 1990.

Formula (2) assumes that the annual interest rate is always the same. What happens when the annual interest rate changes from time to time? We can still define the accumulation factor $A(t_1,t_2)$ by (2) if the annual rate of interest remains constant between t_1 and t_2. Moreover, we can link the accumulations in periods with different annual interest rates by (1). That means, if the interval between t_1 and t_2 contains a point t^*, where the annual interest rate changes, $A(t_1,t_2)$ can be defined as $A(t_1,t^*)$ $A(t^*,t_2)$. As a result, we obtained compound interest again. It is illustrated by the following example.

EXAMPLE 1.4. What is the accumulation on October 1, 1992 of a \$2000 deposit made on June 1, 1988, if the annual rate of interest is 4.5% in 1988, 4% in 1989, 3% in 1990 and 1991, and 4% in 1992? A compound interest is used, satisfying (2) in each calendar year.

Solution: First, we find that the number of days from June 1 to January 1 is 214, and from January 1 to October 1 is 273. Thus, the accumulation factor for the interval from June 1 to the end of 1988 is $(1.045)^{\frac{214}{365}}$. The accumulation factor for the next three years is $1.04(1.03)^2$, and for the

period from the beginning to October 1 of 1992 it is $(1.04)^{\frac{273}{365}}$. Therefore the accumulation factor for the whole term is

$$(1.045)^{\frac{214}{365}}\ 1.04(1.03)^2\ (1.04)^{\frac{273}{365}} = 1.16589.$$

So the accumulated value is $\$2000 \times 1.16589 = \$2331.78$.

Let us return to the general accumulation factor $A(t_1,t_2)$. If we want to work with it, we have to know the values of a function in two variables. However, if we have a compound interest, $A(t_1,t_2)$ can be expressed in a simpler form. In fact, we can write

$$A(t_1,t_2) = \frac{A(t_0,t_2)}{A(t_0,t_1)}, \text{ where } t_0 < t_1 < t_2.$$

Thus, if we fix t_0 and define the function $w_{t_0}(t) = A(t_0,t)$ then

$$A(t_1,t_2) = \frac{w_{t_0}(t_2)}{w_{t_0}(t_1)}.$$

Hence, $w_{t_0}(t)$ makes it possible to find the values of $A(t_1,t_2)$ for any $t_0 < t_1 < t_2$. The function $w_{t_0}(t)$ has only one argument but it still depends on t_0. How can we get rid of t_0? If we pick another number, say t_0^*, where $t_0^* < t_0$, then

$$A(t_0^*,t) = A(t_0^*,t_0)\, A(t_0,t).$$

If we take the logarithm of both sides of the equation, and then differentiate with respect to t, we get

$$\frac{d}{dt} \log A(t_0^*,t) = \frac{d}{dt} \log A(t_0,t).$$

Note that log denotes the natural logarithm; that is, the logarithm to the base e. Therefore, the function $\frac{d}{dt} \log A(t_0,t)$ does not depend on the special choice of t_0 any more. Let us denote it by $\delta(t)$:

$$\begin{aligned}\delta(t) &= \frac{d}{dt} \log A(t_0,t) \\ &= \frac{1}{A(t_0,t)} \frac{d}{dt} A(t_0,t). \qquad (3)\end{aligned}$$

The function $\delta(t)$ is called the *force of interest per annum.* It is already a function in one variable. Using the fundamental theorem of calculus, we have

$$\begin{aligned}
\int_{t_1}^{t_2} \delta(t)dt &= \int_{t_1}^{t_2} \frac{d}{dt} \log A(t_0,t)dt \\
&= \log A(t_0,t_2) - \log A(t_0,t_1) \\
&= \log \frac{A(t_0,t_2)}{A(t_0,t_1)} \\
&= \log A(t_1,t_2).
\end{aligned}$$

Therefore,

$$A(t_1,t_2) = e^{\int_{t_2}^{t_1} \delta(t)dt}. \tag{4}$$

If we want to be mathematically precise, we must require $A(t_0,t)$ to satisfy certain regularity conditions in order for the above reasoning to be correct. For example, if $A(t_0,t)$ has a continuous derivative with respect to t, then $\delta(t)$ exists and is continuous.

So far we have proved that if (1) holds, then $A(t_1,t_2)$ can be expressed by (4). The converse is also true. That means if the function $\delta(t)$ is given, then defining $A(t_1,t_2)$ by (4), equation (1) is satisfied. In fact, we get

$$\begin{aligned}
A(t_1,t_2)\, A(t_2,t_3) &= e^{\int_{t_1}^{t_2} \delta(t)dt}\, e^{\int_{t_2}^{t_3} \delta(t)dt} \\
&= e^{\int_{t_1}^{t_2} \delta(t)dt + \int_{t_2}^{t_3} \delta(t)dt} \\
&= e^{\int_{t_1}^{t_3} \delta(t)dt} \\
&= A(t_1,t_3).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\frac{d}{dt} A(t_0,t) &= \frac{d}{dt} \int_{t_0}^{t} \delta(s)ds \\
&= \delta(t).
\end{aligned}$$

Again, mathematical correctness requires that $\delta(t)$ satisfy a regularity condition. For example, if $\delta(t)$ is continuous, then the definition of $A(t_1,t_2)$ by (4) is correct, and $A(t_0,t)$ is continuously differentiable in t.

We have proven a mathematical result that will be needed in later chapters. Therefore, we state it in the following theorem.

THEOREM 1.1. *Let the function $f(t_1,t_2)$ be positive and continuously differentiable in t_2. Then it satisfies*

$$f(t_1,t_2)\, f(t_2,t_3) = f(t_1,t_3)$$

if and only if $f(t_1,t_2)$ can be expressed as

$$f(t_1,t_2) = e^{\int_{t_1}^{t_2} g(t)dt},$$

where $g(t)$ is continuous. Also, the relationship between f and g is

$$\begin{aligned} g(t) &= \frac{d}{dt} \log f(t_0,t) \\ &= \frac{1}{f(t_0,t)} \frac{d}{dt} f(t_0,t) \end{aligned}$$

where t_0 can be any number less than t. ■

We defined $\delta(t)$ in a rather abstract way. However, it also has an interpretation that can be appreciated more easily.

Let t_1 and t_2 be very close to each other. Then

$$\begin{aligned} A(t_1,t_2) &= e^{\int_{t_1}^{t_2} \delta(t)dt} \\ &\approx e^{(t_2-t_1)\delta(t_1)}, \end{aligned}$$

where "≈" means approximately equal. We also know from calculus that if the number h is very small, then $e^h \approx 1 + h$.

Now, if t_2 is close to t_1, then $t_2 - t_1$ and $(t_2 - t_1)\,\delta(t_1)$ are small. Consequently,

$$e^{(t_2-t_1)\delta(t_1)} \approx 1 + (t_1 - t_1)\,\delta(t_1),$$

and

$$A(t_1,t_2) \approx 1 + (t_2 - t_1)\, \delta(t_1). \tag{5}$$

That means if we invest \$1 at time t_1 then after a very short time period, say h, our money will accumulate to approximately $1 + h\delta(t_1)$.

Let us see next what happens when $\delta(t)$ takes on a special form: $\delta(t) = \delta$; that is, $\delta(t)$ is the constant function. Then,

$$\begin{aligned} A(t_1,t_2) &= e^{\int_{t_1}^{t_2} \delta(t)dt} \\ &= e^{(t_2-t_1)\delta}. \end{aligned} \tag{6}$$

If $t_2 = t_1 + 1$; that is, t_2 is exactly one year after t_1, we get

$$A(t_1,t_1 + 1) = e^{\delta}.$$

On the other hand, since $A(t_1,t_1 + 1)$ is the accumulated value of 1 one year after the time of the deposit t_1, it equals the capital 1 plus the interest

$$A(t_1,t_1 + 1) = 1 + i.$$

Therefore

$$1 + i = e^{\delta} \tag{7}$$

and since δ does not depend on t, neither does i.

As a result, (7) gives the relationship between the constant force of interest per annum: δ, and the constant annual rate of interest: i.

We can pick other special values for t_2. Let $t_2 = t_1 + \frac{1}{2}$. That means we are interested in the accumulation over a half year. Then

$$A\left(t_1,\, t_1 + \frac{1}{2}\right) = e^{\frac{1}{2}\delta} = (1 + i)^{\frac{1}{2}}.$$

If $t_2 = t_1 + \frac{1}{12}$; that is, the accumulation is computed for one month, we get

$$A\left(t_1,\, t_1 + \frac{1}{12}\right) = e^{\frac{1}{12}\delta} = (1 + i)^{\frac{1}{12}}.$$

Furthermore, $t_2 = t_1 + n$ (n integer) indicates the accumulated value is to be determined for n years. Then we have

$$A(t_1, t_1 + n) = e^{n\delta} = (1 + i)^n.$$

In general, from (7) we get

$$A(t_1,t_2) = e^{(t_2-t_1)\delta} = (1 + i)^{t_2-t_1}.$$

Thus (2) means we have a constant force of interest per annum.

EXAMPLE 1.5. Let the force of interest per annum be 0.05. If we invest \$4000 on March 1, 1990, how much will be its accumulated value on

a) April 1, 1990?

b) September 20, 1990?

c) March 1, 1993?

What is the annual rate of interest?

Solution: We have $\delta = 0.05$.

a) The number of days between March 1 and April 1 is 31. So the accumulated value is

$$\$4000\, e^{\frac{31}{365}(0.05)} = \$4000 \times 1.00426 = \$4017.04.$$

b) The number of days between March 1 and September 20 is $31 + 30 + 31 + 30 + 31 + 31 + 19 = 203$. So, the accumulated value is

$$\$4000\, e^{\frac{203}{365}(0.05)} = \$4000 \times 1.0282 = \$4112.80.$$

c) There are exactly three years between the deposit and the withdrawal, so the accumulated value is

$$\$4000\, e^{3(0.05)} = \$4000 \times 1.116183 = \$4647.32.$$

Using (7) the annual rate of interest is

$$\begin{aligned} i &= e^{\delta} - 1 \\ &= e^{0.05} - 1 \\ &= 1.05127 - 1 \\ &= 0.05127, \end{aligned}$$

or equivalently, $i = 5.127\%$.

In this example we found that $\delta = 0.05$ results in an annual interest rate of 0.05127. It is always true that the annual rate of interest is higher than the force of interest per annum. This can be seen from the Taylor expansion of e^δ:

$$e^\delta = 1 + \delta + \frac{\delta^2}{2!} + \frac{\delta^3}{3!} + \ldots$$

because then

$$\begin{aligned} i &= e^\delta - 1 \\ &= \delta + \frac{\delta^2}{2!} + \frac{\delta^3}{3!} + \ldots > \delta. \end{aligned}$$

However, as we have already mentioned, e^δ is close to $1 + \delta$ if δ is small. Therefore, i is close to δ for small δ. Let us see some examples.

δ	i
0.01	0.01005
0.02	0.02020
0.03	0.03045
0.06	0.06184
0.10	0.10517
0.15	0.16183
0.20	0.22140

Let us recall that the effective rate of interest for the period between t_1 and t_2 was defined as

$$i_{eff}(t_1,t_2) = A(t_1,t_2) - 1,$$

where $A(t_1,t_2)$ is the accumulation at t_2 of \$1 invested at t_1. We often use another type of interest, which is called the nominal rate of interest. The *nominal rate of interest* per annum is defined as the number $i_{nom}(t_1,t_2)$ satisfying $i_{eff}(t_1,t_2) = (t_2 - t_1)\, i_{nom}(t_1,t_2)$. Note that if we are using simple interest, the nominal rate of interest per annum is equal to the effective interest rate per annum, but for compound interest this is usually not true. Later on we will see that the nominal rate of interest per annum is less than the effective interest rate per annum if the interest is defined by (2) and $t_2 - t_1 < 1$.

If we use the expression interest rate without adding the word effective or nominal, it will always mean an effective rate of interest.

Using these definitions, we can give another interpretation to the force of interest. Since

$$i_{eff}(t_1,t_2) = A(t_1,t_2) - 1,$$

we get

$$i_{nom}(t_1,t_2) = \frac{A(t_1,t_2) - 1}{t_2 - t_1}.$$

If we recall (5), we see that as t_2 approaches t_1, $i_{nom}(t_1,t_2)$ tends to $\delta(t_1)$. Hence at any point t, the force of interest per annum is the limit of the nominal rate of interest per annum as the length of the term goes to zero.

If the interest satisfies (2), the effective and nominal rates of interest depend on t_1 and t_2 only through $t_2 - t_1$. Therefore, we can use the notation $i(h)$ instead of $i_{eff}(t_1, t_1 + h)$, and the notation i_h instead of $i_{nom}(t_1, t_1 + h)$. Then we have

$$i(h) = (1 + i)^h - 1 \tag{8}$$

and

$$i_h = \frac{i(h)}{h} = \frac{(1 + i)^h - 1}{h}. \tag{9}$$

EXAMPLE 1.6. Let the effective annual rate of interest be 4%. Find the effective rates of interest and the corresponding nominal rates of interest per annum for the following periods

a) January 1 to October 1.

b) January 1 to March 1.

c) January 1 to January 15.

What is the force of interest per annum?

Solution: a) There are 273 days between January 1 and October 1, so $h = \frac{273}{365}$ and the effective rate of interest is

$$\begin{aligned} i(h) &= (1.04)^{\frac{273}{365}} - 1 \\ &= 0.02977. \end{aligned}$$

The nominal rate of interest per annum is

$$\begin{aligned} i_h &= \frac{0.02977}{\frac{273}{365}} \\ &= 0.03980. \end{aligned}$$

b) There are 59 days between January 1 and October 1, so $h = \frac{59}{365}$. As a result, the effective rate of interest is

$$i(h) = (1.04)^{\frac{59}{365}} - 1$$
$$= 0.00636$$

and the nominal rate of interest per annum is

$$i_h = \frac{0.006360}{\frac{59}{365}}$$
$$= 0.03935.$$

c) There are 14 days between January 1 and January 15, so $h = \frac{14}{365}$. The effective rate of interest is

$$i(h) = (1.04)^{\frac{14}{365}} - 1$$
$$= 0.00151$$

and the nominal rate of interest per annum is

$$i_h = \frac{0.00151}{\frac{14}{365}}$$
$$= 0.03925.$$

The force of interest per annum can be obtained by using (7)

$$\delta = \log(1 + i)$$
$$= \log 1.04$$
$$= 0.03922.$$

We can clearly see that as h goes to zero, i_h tends to δ.

Note that in this example, the nominal rate of interest per annum is decreasing as h is decreasing. This is always true, shown in the following theorem.

THEOREM 1.2. *If $A(t_1,t_2) = (1 + i)^{t_2 - t_1}$ is satisfied then the nominal rate of interest per annum i_h is monotone increasing in h and approaches δ as h goes to zero.*

Proof: Let us write (9) as

$$i_h = \frac{(1+i)^h - 1}{h}$$
$$= \frac{(1+i)^h - (1+i)^0}{h - 0}. \tag{10}$$

This is the difference quotient of the function

$$f(x) = (1+i)^x. \tag{11}$$

Therefore, i_h is the slope of the line segment joining $(0,f(0))$ and $(h,f(h))$. Since the function $f(x) = (1+i)^x$ is concave up, i_h is increasing in h.

If h tends to zero, $\frac{f(h) - f(0)}{h - 0}$ goes to $f'(0)$. But when $f'(x) = \log(1+i)\,(1+i)^x$, $f'(0) = \log(1+i) = \delta$ because of (7). ■

From the theorem, it immediately follows that $i_h < i$, if $h < 1$. Hence the nominal rate of interest per annum for a term shorter than a year is less than the effective annual rate of interest.

If $h = \frac{1}{p}$, where p is a positive integer; that is, the term of the transaction is one p^{th} of a year, i_h has one more notation: $i^{(p)}$. Using (9), we have

$$i^{(p)} = \frac{(1+i)^{\frac{1}{p}} - 1}{\frac{1}{p}}$$
$$= p((1+i)^{\frac{1}{p}} - 1), \tag{12}$$

or equivalently

$$\left(1 + \frac{i^{(p)}}{p}\right)^p = 1 + i. \tag{13}$$

We call $i^{(p)}$ the *nominal rate of interest* per annum convertible $p$thly. That means, the interest is payable $p$thly.

If we take $p = 12$ in (12), we get $i^{(12)}$. Although one month is not exactly one twelfth of a year, $\frac{i^{(12)}}{12}$ can be regarded as the *effective rate of*

interest per month. Similarly, $\frac{i^{(4)}}{4}$ is the effective rate of interest per quarter and $\frac{i^{(2)}}{2}$ is the effective rate of interest per half year. Obviously, $\frac{i^{(365)}}{365}$ gives the exact effective rate of interest per day, unless we are dealing with a leap year. If an amount is invested for one day only, it is also called *overnight money*.

EXAMPLE 1.7. Let $i = 0.04$. Determine $i^{(2)}, i^{(4)}, i^{(12)}$, and $i^{(365)}$.
Solution: From (12) we get

$$i^{(2)} = \frac{(1.04)^{\frac{1}{2}} - 1}{\frac{1}{2}} = 0.03961,$$

$$i^{(4)} = \frac{(1.04)^{\frac{1}{4}} - 1}{\frac{1}{4}} = 0.03941,$$

$$i^{(12)} = \frac{(1.04)^{\frac{1}{12}} - 1}{\frac{1}{12}} = 0.03928,$$

and

$$i^{(365)} = \frac{(1.04)^{\frac{1}{365}} - 1}{\frac{1}{365}} = 0.03922.$$

We can see that as p gets larger, $i^{(p)}$ tends to $\delta = \log 1.04 = 0.03922$. This is due to the fact that as p increases, $h = \frac{1}{p}$ tends to zero.

EXAMPLE 1.8. A bank uses the following nominal interest rates: 6% per annum convertible yearly, 5.5% per annum convertible monthly, and 5% per annum convertible daily. Find the accumulation of \$3000 if it is invested for

a) one day.
b) one month.

c) one year.

Solution: a) The accumulation in one day is

$$3000\left(1+\frac{0.05}{365}\right)=\$3000.41.$$

b) The accumulation in one month is

$$3000\left(1+\frac{0.055}{12}\right)=\$3013.75.$$

c) The accumulation in one year is

$$3000\,(1+0.06)=\$3180.$$

Next, we want to examine the different ways in which compound interest can be paid to the investor. Assume an investor invests \$1 at time t_0 and wants to get back this capital of \$1 at time t_e. The most obvious way of paying interest is the payment of $i_{eff}(t_0,t_e)$ at time t_e.

However, there are other possibilities. Let us pick $n - 1$ points in the time interval between t_0 and t_e, say $t_1,t_2,\ldots,t_{n-1}$ so that $t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = t_e$. When a capital of \$1 is invested at time t_0, it will earn an interest of $i_{eff}(t_0,t_1)$ by t_1. If this interest is paid back to the investor at t_1, the outstanding capital is still \$1. This will earn interest of $i_{eff}(t_1,t_2)$ by t_2 and this interest is again paid back to the investor. The amount of the capital drops again to \$1. If we repeat this procedure n times we get the following sequence of interest payments:

$$i_{eff}(t_0,t_1) \text{ at } t_1, \tag{14}$$

$$i_{eff}(t_1,t_2) \text{ at } t_2, \tag{15}$$

$$\vdots$$

$$i_{eff}(t_{n-1},t_n) \text{ at } t_n. \tag{16}$$

After paying $i_{eff}(t_{n-1},t_n)$ at t_n, the outstanding capital is again \$1 and this can be returned to the investor.

Using nominal rates of interest, we can rewrite (14)-(16) as follows:

$$i_{\text{nom}}(t_0,t_1)\cdot(t_1 - t_0) \text{ at } t_1, \tag{17}$$

$$i_{\text{nom}}(t_1,t_2)\cdot(t_2 - t_1) \text{ at } t_2, \tag{18}$$

$$\vdots$$

$$i_{\text{nom}}(t_{n-1},t_n)\cdot(t_n - t_{n-1}) \text{ at } t_n. \tag{19}$$

In the case of interest rates defined by (2) we can formulate our result in the following theorem.

THEOREM 1.3. *A capital of* \$1 *is invested at an annual interest rate of* i *for* n *years. The interest satisfies* $A(t_1,t_2) = (1 + i)^{t_2-t_1}$. *Then the interest can be paid in any of the following ways:*

a) Payment of interest $(1 + i)^n - 1$ *at the end of year* n.

b) Payment of interest i *at the end of each year.*

c) Payment of interest $\frac{i^{(p)}}{p}$ *at the end of each* $\frac{1}{p}$ *year long period.*

The outstanding capital after each interest payment is \$1. ■

Later on, we will find some more methods of paying interest.

EXAMPLE 1.9. We invest \$300 at an annual rate of interest of 8% for five years.

a) How much interest will we get at the end of the fifth year?

b) If the interest is paid at the end of each year, how much will these payments be?

c) How much will the interest payments be if they are paid monthly?

Solution: a) The interest at the end of the fifth year will be

$$\begin{aligned}\$300((1.08)^5 - 1) &= \$300 \times 0.4693\\ &= \$140.80.\end{aligned}$$

b) The yearly interest payment is

$$\$300 \times 0.08 = \$24.$$

c) First we need to determine $\frac{i^{(12)}}{12}$. From (12) we get

$$\frac{i^{(12)}}{12} = (1.08)^{\frac{1}{12}} - 1.$$

So the monthly interest payments are

$$\begin{aligned}\$300((1.08)^{\frac{1}{12}} - 1) &= \$300 \times 0.00643\\ &= \$1.93.\end{aligned}$$

We can see that the total interest paid in (a), (b), and (c) of Example 1.9 are all different. In (a) it is \$140.80, in (b) it is $\$24 \times 5 = \$120$, and in (c) it is $\$1.93 \times 5 \times 12 = \$115.80$. It is always true that the total amount of interest paid yearly is less than a single interest payment at the end of year n and

the total amount of interest payments made more often than yearly is even less than this. Indeed,

$$ni < (1 + i)^n - 1,$$

since using the binomial theorem, we get

$$(1 + i)^n - 1 = 1 + ni + \binom{n}{2} i^2 + \dots + \binom{n}{n} i^n - 1$$
$$= ni + \binom{n}{2} i^2 + \dots + \binom{n}{n} i^n > ni.$$

Because it has already been proven that

$$i_h < i \text{ for } h < 1,$$

we get

$$i^{(p)} < i \text{ for } p > 1.$$

Therefore,

$$n \cdot p \cdot \frac{i^{(p)}}{p} < ni$$

is also true.

Let us look at (17)–(19) again. If n is large and the differences $t_1 - t_0, t_2 - t_1, \dots, t_e - t_{n-1}$ are small, the expressions (17)–(19) can be approximated by

$$\begin{array}{l} \delta(t_0)(t_1 - t_0) \text{ at } t_1, \\ \delta(t_1)(t_2 - t_1) \text{ at } t_2, \\ \vdots \\ \delta(t_{n-1})(t_n - t_{n-1}) \text{ at } t_n. \end{array}$$

If we denote the total interest paid between t_0 and t by $I(t_0,t)$, we get

$$I(t_0,t_k) \approx \sum_{i=1}^{k} \delta(t_i)(t_i - t_{i-1}) \text{ for } k = 1,2,\dots,n,$$

which is an approximation to

$$\int_{t_0}^{t_k} \delta(s)ds.$$

We get

$$I(t_0,t_k) \approx \int_{t_0}^{t_k} \delta(s)ds \text{ for } k = 1,2,...,n.$$

Consequently, if t_k's are close to each other, the function $D(t_0,t)$, defined by

$$D(t_0,t) = \int_{t_0}^{t} \delta(s)ds \text{ for } t_0 < t < t_e,$$

gives a reasonable approximation to the total interest paid from t_0 to t.

This introduction of the function $D(t_0,t)$ is rather heuristic. Now, we want to derive it in a more correct mathematical way.

First we want to define what we mean by continuous interest payments. In general, a continuous cash flow can be defined by the function $M(t_0,t)$ $(t \geq t_0)$ where t_0 is fixed and the function is continuous in t and $M(t_0,t)$ gives the total payment made from t_0 to t. Obviously, if $t_0 < t_1 < t_2$, then $M(t_0,t_2) - M(t_0,t_1)$ gives the payment made in the time interval from t_1 to t_2. Thus $M(t_0,t+h) - M(t_0,t-h)$ is the payment made from $t-h$ to $t+h$. Since $M(t_0,t)$ is a continuous function in t, $M(t_0,t+h) - M(t_0,t-h)$ approaches zero as h goes to zero. This implies that the payment made exactly at time t is 0. So in this case, it does not make sense to talk about a payment at a given point in time. We have to examine payments in time intervals.

Although $M(t_0,t+h) - M(t_0,t)$ goes to zero as h tends to zero, if $M(t_0,t)$ is differentiable with respect to t, we can define the function

$$\rho(t) = \lim_{h \to 0} \frac{M(t_0,t+h) - M(t_0,t)}{h}.$$

That is, $\rho(t)$ is the derivative of $M(t_0,t)$ with respect to t. So, we can write

$$\rho(t) = \frac{d}{dt} M(t_0,t).$$

Note that the function $\rho(t)$ does not depend on the choice of t_0. Indeed, if we pick another t_0, say t_0^* ($t_0^* > t_0$), and $M(t_0^*,t)$ denotes the payment made from t_0^* to t, then

$$M(t_0^*,t) = M(t_0,t) - M(t_0,t_0^*).$$

Since t_0 and t_0^* are constant with respect to t, we get

$$\frac{d}{dt} M(t_0^*,t) = \frac{d}{dt} M(t_0,t).$$

The function $\rho(t)$ is called the *rate of payment* per annum at time t.

Now, let us assume we want to derive a continuous interest payment stream that leaves the investment unchanged all the time. That means our goal is to define a continuous function $D(t_0,t)$ in t such that $D(t_0,t)$ gives the total interest paid from t_0 to t on an initial deposit of \$1 at time t_0 for every t such that the outstanding capital remains \$1 all the time.

Let us see what happens at time t. The deposit at t is \$1, and $D(t_0,t+h) - D(t_0,t)$ is the interest paid in the time interval from t to $t+h$. If h is small, the \$1 capital at t will accumulate to approximately $1 + \delta(t)h$, because of (5). Since the continuous flow of interest has to keep the investment unchanged, we have

$$D(t_0,t+h) - D(t_0,t) \approx \delta(t) \cdot h$$

$$\frac{D(t_0,t+h) - D(t_0,t)}{h} \approx \delta(t). \tag{20}$$

Taking the limit on both sides of (20), as h tends to zero, we get

$$\frac{d}{dt} D(t_0,t) = \delta(t). \tag{21}$$

Using the fundamental theorem of calculus, from (21) we get the following result.

$$D(t_0,t) = \int_{t_0}^{t} \delta(s)ds. \tag{22}$$

Because of (21), the annual rate of the interest payment at time t is $\delta(t)$.

In the special case when $\delta(t) = \delta$, we get

$$D(t_0,t) = (t - t_0)\delta. \tag{23}$$

Thus we get the following result.

THEOREM 1.4. *A capital of* \$1 *is invested at an annual interest rate of* i *for* n *years. The interest satisfies* $A(t_1,t_2) = (1 + i)^{t_2 - t_1}$. *Then the continuous interest payment stream at an annual rate of* δ *keeps the outstanding capital unchanged* (\$1) *over the whole* n *year long period.* ■

EXAMPLE 1.10. A deposit of \$400 is made on January 1 for 5 years in an account earning 6% interest a year. Assume the interest is paid continuously.

a) What is the annual rate of the interest payment?
b) What is the total amount of the interest payment?
c) What is the interest paid in one month?

Solution: a) The rate of payment per annum is

$$400\,\delta = 400\log(1 + i) = 400\log 1.06 = 400(0.05827) = 23.31.$$

b) The interest paid during the 5 year long period is

$$5(400)\delta = 5(23.31) = \$116.55.$$

c) The interest paid in one month is

$$\frac{1}{12}(400)\delta = \frac{1}{12}(23.31) = \$1.94.$$

Now we discuss some more methods of paying interest.

Consider the following simple situation. We deposit \$1 at time t_1 at an annual rate of interest i and withdraw the accumulated amount at t_2 at an effective rate of interest of $i_{eff}(t_1,t_2)$. Then we will get $1 + i_{eff}(t_1,t_2)$ back at t_2.

Now, assume that we need exactly \$1 at time t_2 so we only want to invest so much money at t_1 that will accumulate to \$1 by t_2. The question is how much the deposit has to be at t_1. Let us denote it by $v(t_1,t_2)$. Then $v(t_1,t_2)\,(1 + i_{eff}(t_1,t_2)) = 1$, hence

$$v(t_1,t_2) = \frac{1}{1 + i_{eff}(t_1,t_2)} = \frac{1}{A(t_1,t_2)}. \qquad (24)$$

The number $v(t_1,t_2)$ is called the discount factor for the time period from t_1 to t_2.

Since an investment of 1 at t_1 accumulates to $1 + i(t_1,t_2)$ by t_2, and an investment of $v(t_1,t_2)$ at t_1 accumulates to 1 by t_2, if we invest $1 - v(t_1,t_2)$ at t_1, its accumulation at t_2 will be $i(t_1,t_2)$. Hence, introducing the notation

$$d_{eff}(t_1,t_2) = 1 - v(t_1,t_2), \tag{25}$$

$d_{eff}(t_1,t_2)$ is the discounted value of $i(t_1,t_2)$. The number $d_{eff}(t_1,t_2)$ is called the effective discount rate for the time period from t_1 to t_2. From (24) and (25) we get

$$d_{eff}(t_1,t_2) = \frac{i_{eff}(t_1,t_2)}{1 + i_{eff}(t_1,t_2)}. \tag{26}$$

Thus, the transaction of investing $v(t_1,t_2)$ at t_1 and withdrawing the accumulation \$1 at t_2 can also be interpreted in the following way.

We invest \$1 at time t_1. An interest of $d_{eff}(t_1,t_2)$ is paid back to us immediately and the capital \$1 is returned at time t_2. Therefore, $d_{eff}(t_1,t_2)$ can also be called an interest paid in advance.

If $t_2 = t_1 + 1$ and omitting t_1 cannot cause any misunderstanding, we usually write d and v instead of $d_{eff}(t_1,t_2)$ and $v(t_1,t_2)$, respectively. Thus we have

$$v = \frac{1}{1+i} \tag{27}$$

$$d = \frac{i}{1+i}. \tag{28}$$

EXAMPLE 1.11. A sum of \$500 is deposited for one year at an annual rate of interest of 3%. How much interest is paid if

a) the interest is paid at the end of the year?

b) the interest is paid at the beginning of the year?

Solution: a) The interest is

$$\$500\, i = \$500(0.03) = \$15.$$

b) The discount rate is

$$d = \frac{i}{1+i} = \frac{0.03}{1.03} = 0.02913.$$

So, the interest paid in advance is

$$\$500\, d = \$14.56.$$

In Example 1.11, we can see that $d < i$. It is always true that

$$d_{eff}(t_1,t_2) < i_{eff}(t_1,t_2), \tag{29}$$

since

$$d_{eff}(t_1,t_2) = \frac{i_{eff}(t_1,t_2)}{1 + i_{eff}(t_1,t_2)} < i_{eff}(t_1,t_2).$$

Annual interest rates are almost always less than 100%. However, it can also happen that $i > 100\%$; for example, in countries with a hyper inflation. On the other hand, d is always less than 1 since

$$d_{eff}(t_1,t_2) = 1 - v(t_1,t_2) < 1.$$

EXAMPLE 1.12. We can invest money at an annual rate of interest of 4% for one year. How much do we have to invest if we want the accumulation to be \$500 after one year?

Solution: The investment has to be

$$\$500\ v.$$

Since $v = \frac{1}{1+i} = \frac{1}{1.04} = 0.96154$, the investment is

$$\$500\ v = \$480.77.$$

What we said about discount factors and discount rates so far does not depend on the type of interest we are working with. However, we can examine these concepts in the context of different types of interests.

First assume we have a simple interest. Let the annual rate of interest be i. Then

$$i(t_1,t_2) = (t_2 - t_1)i,$$

and (26) gives

$$d(t_1,t_2) = (t_2 - t_1)\frac{i}{1 + (t_2 - t_1)i}.$$

It is important to note that, in general,

$$d(t_1,t_2) \neq (t_2 - t_1)d,$$

since together with (28), this would imply that

$$\frac{i}{1 + (t_2 - t_1)i} = \frac{i}{1 + i}$$

which is only true if $t_2 - t_1 = 1$.

Next we turn our attention to compound interest. First note that because of (4) and (24), we get

$$v(t_1,t_2) = e^{-\int_{t_1}^{t_2} \delta(t)dt}$$

If we use the principle of consistency and (24), we get

$$v(t_1,t_n) = v(t_1,t_2)\, v(t_2,t_3) \dots v(t_{n-1},t_n) \tag{30}$$

for $t_1 < t_2 < \dots < t_{n-1} < t_n$. From (25) and (30) we obtain

$$1 - d_{eff}(t_1,t_n) = (1 - d_{eff}(t_1,t_2))(1 - d_{eff}(t_2,t_3)) \cdot \dots \cdot (1 - d_{eff}(t_{n-1},t_n)). \tag{31}$$

Equations (30) and (31) can be interpreted as follows.

We want to receive \$1 at time t_n. Then we need to invest $v(t_{n-1},t_n)$ at time t_{n-1}. In order to have $v(t_{n-1},t_n)$ available at time t_{n-1}, we have to invest $v(t_{n-2},t_{n-1})\, v(t_{n-1},t_n)$ at time t_{n-2}, etc. Finally, we must invest $v(t_1,t_2) \dots v(t_{n-2},t_{n-1})\, v(t_{n-1},t_n)$ at time t_1. On the other hand, we also know that an investment of $v(t_1,t_n)$ at time t_1 will accumulate to \$1 by time t_n.

Finally, assume we have a compound interest of the form (2). Then, (26) implies

$$\begin{aligned} 1 - d_{eff}(t_1,t_2) &= \frac{1}{1 + i_{eff}(t_1,t_2)} \\ &= \frac{1}{A(t_1,t_2)} \\ &= \frac{1}{(1+i)^{t_2-t_1}} \\ &= \left(\frac{1}{1+i}\right)^{t_2-t_1} \\ &= (1-d)^{t_2-t_1}. \end{aligned} \tag{32}$$

As a result,

$$d_{eff}(t_1,t_2) = 1 - (1-d)^{t_2-t_1}. \tag{33}$$

Again, it is worth remembering that, in general,

$$d_{eff}(t_1,t_2) \neq d^{t_2-t_1}.$$

Indeed, combining this equation with (33) we would get

$$1 - (1 - d)^{t_2 - t_1} = d^{t_2 - t_1} \quad \text{for } t_2 > t_1$$

or

$$d^{t_2 - t_1} + (1 - d)^{t_2 - t_1} = 1.$$

Introducing $h = t_2 - t_1$, this can be written as

$$d^h + (1 - d)^h = 1, \text{ for } h > 0.$$

However, $0 < d < 1$ and d^h and $(1 - d)^h$ are both monotone decreasing functions in h, so their sum can take on the value 1 only for one h. Now, $h = 1$ satisfies the equation, and hence

$$d_{eff}(t_1, t_2) = d^{t_2 - t_1},$$

only if $t_2 = t_1 + 1$.

Also note, that because of (33), $d_{eff}(t_1, t_2)$ only depends on t_1 and t_2 through $t_2 - t_1$. So, we can introduce the notation

$$d(h) = d_{eff}(t_1, t_1 + h).$$

Then from (33), we get

$$d(h) = 1 - (1 - d)^h. \tag{34}$$

We can also talk about nominal discount rate per annum which is defined by

$$d_{eff}(t_1, t_2) = (t_2 - t_1)\, d_{nom}(t_1, t_2).$$

Dividing both sides of (29) by $(t_2 - t_1)$, we get

$$d_{nom}(t_1, t_2) < i_{nom}(t_1, t_2).$$

If we have a compound interest rate, $d_{nom}(t_1, t_2)$ approaches $\delta(t_1)$ as t_2 goes to t_1. Indeed, if we divide both sides of (26) by $(t_2 - t_1)$, we obtain

$$\begin{aligned} d_{nom}(t_1, t_2) &= \frac{i_{nom}(t_1, t_2)}{1 + i_{eff}(t_1, t_2)} \\ &= \frac{i_{nom}(t_1, t_2)}{1 + i_{nom}(t_1, t_2)(t_2 - t_1)}. \end{aligned} \tag{35}$$

Now, if t_2 goes to t_1, $i_{nom}(t_1, t_2)$ tends to $\delta(t_1)$, as we have already proven and $t_2 - t_1$ goes to 0. So, $d_{nom}(t_1, t_2)$ approaches $\delta(t_1)$.

If the interest satisfies (2), we use the notation

$$d_h = d_{nom}(t_1, t_1 + h).$$

Then (34) gives

$$d_h = \frac{d(h)}{h} = \frac{1 - (1 - d)^h}{h}. \tag{36}$$

EXAMPLE 1.13. Let the annual rate of interest be 4%. Find the effective discount rates and the nominal discount rates per annum for the following time periods:

a) January 1 to October 1.
b) January 1 to March 1.
c) January 1 to January 15.

Solution: The annual discount rate can be obtained from (28).

$$d = \frac{i}{1 + i} = \frac{0.04}{1.04} = 0.03846.$$

a) There are 273 days between January 1 and October 1, so $h = \frac{273}{365}$. As a result, the effective discount rate is

$$\begin{aligned} d(h) &= 1 - (1 - 0.03846)^{\frac{273}{365}} \\ &= 0.02891 \end{aligned}$$

and the nominal discount rate per annum is

$$d_h = \frac{0.02891}{\frac{273}{365}} = 0.03865.$$

b) There are 59 days between January 1 and March 1. So, $h = \frac{59}{365}$. Hence, the effective discount rate is

$$d(h) = 1 - (1 - 0.03846)^{\frac{59}{365}} = 0.00632$$

and the nominal discount rate per annum is

$$d_h = \frac{0.00632}{\frac{59}{365}} = 0.03910.$$

c) There are 14 days between January 1 and January 15 and $h = \frac{14}{365}$. Therefore, the effective discount rate is

$$d(h) = 1 - (1 - 0.03846)^{\frac{14}{365}} = 0.00150$$

and the nominal discount rate per annum is

$$d_h = \frac{0.00150}{\frac{14}{365}} = 0.03911.$$

In this example, the nominal discount rate is increasing as h is decreasing. We can also see that the nominal discount rate goes to $\delta = 0.03922$ as h tends to zero. Again, we can state a general theorem.

THEOREM 1.5. *If* $A(t_1,t_2) = (1 + i)^{t_2 - t_1}$ *is satisfied then the nominal discount rate per annum* d_h *is monotone decreasing in* h *and* d_h *approaches* δ *as* h *goes to zero.*

Proof: Let us rewrite (36) as

$$d_h = \frac{1 - (1 - d)^h}{h} = -\frac{(1 - d)^h - (1 - d)^0}{h - 0}.$$

So $-d_h$ is the difference quotient of the function $g(x) = (1 - d)^x$, or in other words, it is the slope of the line segment joining $(0,g(0))$ and $(h,g(h))$. The function $g(x) = (1 - d)^x$ is concave up, thus $-d_h$ is increasing in h. Therefore, d_h is decreasing in h.

If h tends to zero, $-\frac{g(h) - g(0)}{h - 0}$ goes to $-g'(0)$. But $g'(x) = \log(1 - d)\,(1 - d)^x$, so

$$-g'(0) = -\log(1 - d).$$

Now,

$$\begin{aligned} -\log(1 - d) &= \log\left(\frac{1}{1 - d}\right) \\ &= \log(1 + i) \\ &= \delta. \ \blacksquare \end{aligned}$$

As a special case of the above theorem, we get $d_h > d$ for $h < 1$. Consequently, the nominal discount rate per annum for a term shorter than a year is greater than the annual discount rate.

If $h = \frac{1}{p}$, where p is a positive integer; that is, the term of the transaction is one p^{th} of a year, d_h has one more notation: $d^{(p)}$. Using (36), we get

$$d^{(p)} = p\,((1 - (1 - d)^{\frac{1}{p}}) \tag{37}$$

or equivalently,

$$\left(1 - \frac{d^{(p)}}{p}\right)^p = 1 - d.$$

We say that $d^{(p)}$ is the nominal discount rate per annum convertible $p$thly.

If we take $n = 12$ in (37), $d^{(12)}$ gives the nominal discount rate per annum convertible monthly. Similarly, $d^{(2)}, d^{(4)}$, and $d^{(365)}$ are the nominal discount rates per annum, convertible half yearly, quarterly, and daily, respectively.

EXAMPLE 1.14. Let $d = 0.03$. Determine $d^{(2)}, d^{(4)}, d^{(12)}$, and $d^{(365)}$.
Solution: Using (37) we get

$$d^{(2)} = 2(1 - (1 - 0.03)^{\frac{1}{2}}) = 0.030228,$$
$$d^{(4)} = 4(1 - (1 - 0.03)^{\frac{1}{4}}) = 0.030344,$$
$$d^{(12)} = 12(1 - (1 - 0.03)^{\frac{1}{12}}) = 0.030421,$$

and

$$d^{(365)} = 365(1 - (1 - 0.03)^{\frac{1}{365}}) = 0.030458.$$

Note that as p increases, $h = \frac{1}{p}$ decreases, so $d^{(p)}$ goes to $\delta = -\log(1 - d) = 0.030459$.

Since paying $i_{eff}(t_1,t_2)$ interest at time t_2 is equivalent to paying $d_{eff}(t_1,t_2)$ at time t_1, using Theorem 1.3 we get the following result.

THEOREM 1.6. *A capital of* $1 *is invested at an annual interest rate of* i *for* n *years. The interest satisfies* $A(t_1,t_2) = (1+i)^{t_2-t_1}$. *Then, the interest can be paid in the following ways.*

a) Payment of $1-(1-d)^n$ *at the beginning of the first year.*

b) Payment of d *at the beginning of each year.*

c) Payment of $\frac{d^{(p)}}{p}$ *at the beginning of each* $\frac{1}{p}$ *year long period.*

The outstanding capital at the end of each payment period is $1. ■

From Theorem 1.3, 1.4, and 1.6 we can get the following result.

THEOREM 1.7. *A capital of* $1 *is invested at an annual interest rate of* i *for* n *years. The interest satisfies* $A(t_1,t_2) = (1+i)^{t_2-t_1}$. *Then, the interest can be paid in the following ways.*

a) Payment of $(1+i)^n - 1$ *at the end of year* n.

b) Payment of $1-(1-d)^n$ *at the beginning of the first year.*

c) Payment of i *at the end of each year.*

d) Payment of d *at the beginning of each year.*

e) Payment of $\frac{i^{(p)}}{p}$ *at the end of each* $\frac{1}{p}$ *year long period.*

f) Payment of $\frac{d^{(p)}}{p}$ *at the beginning of each* $\frac{1}{p}$ *year long period.*

g) Continuous payment stream at an annual rate of δ.

The outstanding capital at the end of each payment period (or at any time in g)) is $1. *Also,*

$$d < d^{(p)} < \delta < i^{(p)} < i, \text{ for any } p > 1 \text{ integer,}$$

and $d^{(p)}$ *and* $i^{(p)}$ *tend to* δ *as* p *goes to infinity.* ■

In the remainder of the book, we assume that we are working with compound interest.

PROBLEMS

1.1. A sum of $500 is deposited in a bank account on February 1, 1989. If the account earns interest at a rate of 4% per annum, what is the accumulated value of the account on February 1, 1990? Find the capital and the interest contents of the accumulation, too.

1.2. A sum of \$2000 is deposited in a bank account on March 1, 1991 at a 3% annual interest rate. Using simple interest determine

a) the accumulated value of the account on May 1, 1991 and the interest paid for the period between March 1 and May 1.
b) the accumulated value of the account on November 1, 1991 and the interest paid for the period between March 1 and November 1.

1.3. A sum of \$800 is deposited in a bank account on May 1, 1988 at a 4% annual rate of interest. Assuming the accumulation factor satisfies (2), find the accumulation on September 1, 1988. What is the interest earned during this period?

1.4. A sum of \$1500 is deposited in a bank account on January 1, 1989. The annual rate of interest is 5%.

a) Based on a compound interest satisfying (2), what is the accumulated value on June 1, 1989? If the accumulation is withdrawn and redeposited immediately afterwards, how large will the account grow by January 1, 1990?
b) Answer the same questions as in part (a), but this time use a simple interest.

1.5. A sum of \$3000 is deposited on March 1, 1988. If the annual rate of interest is 4% in 1988 and 1989, 5% in 1990, 3% in 1991, and 3.5% in 1992 and 1993, and the a compound interest satisfies (2) in each calendar year, determine the accumulation on March 1, 1993.

In the remaining problems of this section, we assume a compound interest rate satisfying (2).

1.6. An interest earning account has a constant force of interest per annum of 0.06. If \$2500 is deposited on January 1, 1991, what is the accumulation on

a) February 1, 1991?
b) June 14, 1991?
c) November 1, 1991?

1.7. Given the following forces of interest per annum, find the corresponding annual rates of interest

a) $\delta = 0.008$.
b) $\delta = 0.025$.
c) $\delta = 0.037$.

1.8. Given the following annual interest rates, find the corresponding forces of interest per annum

a) $i = 0.025$.
b) $i = 0.05$.
c) $i = 0.07$.

1.9. Using a 5% effective annual rate of interest, determine the effective rates of interest and the corresponding nominal rates of interest per annum for the following periods

a) March 1 to April 1.
b) March 1 to September 1.
c) March 1 to December 17. Also find the force of interest per annum.

1.10. Let $i = 0.05$. Find $i^{(2)}, i^{(3)}, i^{(6)}$, and $i^{(365)}$.

1.11. Assume the following nominal rates of interest are used:

7% per annum convertible yearly,
6.7% per annum convertible weekly,
6.4% per annum convertible daily.

Determine the accumulation of \$2500 if it is invested for

a) one day.
b) one week.
c) one year.

1.12. A sum of \$2000 is invested at a 7% annual rate of interest for four years.

a) How much interest is paid at the end of year four?
b) Determine the interest payments if they are made at the end of each year.
c) If the interest is paid monthly, find its monthly amount.

1.13. The interest on a \$500 deposit is paid continuously for 3 years. Assume the annual rate of interest is 7%.

a) Determine the annual rate of the interest payment.
b) Find the total amount of the interest payment.
c) Obtain the interest paid in one quarter.

1.14. a) If $i = 0.04$, find d, δ, and v.
b) If $d = 0.03$, find i, δ, and v.
c) If $\delta = 0.035$, find i, d, and v.
d) If $v = 0.97$, find i, d, and δ.

1.15. A sum of \$800 is deposited for one year. Based on a 4% annual rate of interest, determine

a) the interest if it is paid at the end of the year.
b) the interest if it is paid at the beginning of the year.

1.16. How much money will accumulate to \$1200 in one year, if a 5% annual rate of interest is used?

1.17. Assuming a 5% annual rate of interest, find the effective discount rates and the nominal discount rates per annum for the following time periods:

a) February 1 to April 1.
b) March 1 to June 1.
c) March 1 to July 12.

1.18. If $d = 0.04$, determine $d^{(2)}, d^{(3)}, d^{(12)}$, and $d^{(365)}$.

1.2. PRESENT VALUE

We have already seen in Section 1.1 that if $t_1 < t_2$, then an amount of $v(t_1,t_2) = \dfrac{1}{A(t_1,t_2)}$ invested at time t_1 will accumulate to \$1 by time t_2. We say that $v(t_1,t_2)$ is the present value of \$1 at time t_1. Obviously, the present value of C is $C\,v(t_1,t_2)$.

Next assume we want to make an investment at time t_0 so that we will get payments of $C_1,C_2,\ldots,C_n$ at times $t_1,t_2,\ldots,t_n$, respectively ($t_0 \le t_1 \le t_2 \le \ldots \le t_n$). What should be the investment at t_0? Denoting it by PV_{t_0} we get

$$PV_{t_0} = C_1\,v(t_0,t_1) + C_2\,v(t_0,t_2) + \ldots + C_n\,v(t_0,t_n). \qquad (1)$$

The notation PV stands for present value. If t_0 is clear from the context, we can omit it from the subscript of PV.

EXAMPLE 2.1. How much money has to be deposited in an account earning interest at a 5% annual rate on January 1, 1988, if we want to make a

withdrawal of \$1000 on January 1, 1989, \$2000 on January 1, 1991, and \$500 on January 1, 1993?

Solution: The deposit can be obtained as the present value

$$PV = \$1000\ \frac{1}{1.05} + \$2000\left(\frac{1}{1.05}\right)^3 + \$500\left(\frac{1}{1.05}\right)^5 = \$3071.82.$$

Now, assume we want to make the same withdrawals as in Example 2.1, but we also make three additional deposits: \$900 on January 1, 1990, \$100 on January 1, 1991, and \$200 on January 1, 1992. We can ask again how much the initial deposit should be on January 1, 1988. In this case, equate the present value of the deposits to the present value of the withdrawals. Denoting the initial deposit on January 1, 1988 by I, we get the equation

$$I + 900\left(\frac{1}{1.05}\right)^2 + 100\left(\frac{1}{1.05}\right)^3 + 200\left(\frac{1}{1.05}\right)^4$$
$$= 1000\left(\frac{1}{1.05}\right) + 2000\left(\frac{1}{1.05}\right)^3 + 500\left(\frac{1}{1.05}\right)^5.$$

Thus, the initial deposit can be expressed in the form of (1) as the present value of a cash flow

$$PV = 1000\frac{1}{1.05} - 900\left(\frac{1}{1.05}\right)^2 + 2000\left(\frac{1}{1.05}\right)^3 - 100\left(\frac{1}{1.05}\right)^3$$
$$- 200\left(\frac{1}{1.05}\right)^4 + 500\left(\frac{1}{1.05}\right)^5 = \$2004.57.$$

This shows that (1) also makes sense if some of the amounts $C_1, C_2, \ldots, C_n$ are negative.

Thus, in general, assume sums of $C_1, C_2, \ldots, C_n$ are due at times $t_1, t_2, \ldots, t_n$, respectively, where $t_1 \le t_2 \le \ldots \le t_n$. The amounts C_i can be negative. Then the present value of this discrete cash flow at t_0, where $t_0 \le t_1$ is

$$PV_{t_0} = C_1\ v(t_0, t_1) + C_2\ v(t_0, t_2) + \ldots + C_n\ v(t_0, t_n).$$

Note that there is a simple relationship between the present values of a given cash flow at different times. Let us take a t_0^* such that $t_0^* < t_0$. Then, the present value at t_0^* is

$$PV_{t_0^*} = C_1\ v(t_0^*, t_1) + C_2\ v(t_0^*, t_2) + \ldots + C_n\ v(t_0^*, t_n).$$

However,

$$v(t_0^*,t_n) = v(t_0^*,t_0)\, v(t_0,t_i) \quad \text{for every } i = 1,2,\ldots,n,$$

and therefore

$$PV_{t_0^*} = v(t_0^*,t_0)\, PV_{t_0}. \tag{2}$$

That is, the present value of the cash flow at t_0^* equals the present value at t_0 times the discount factor for the time period between t_0^* and t_0.

EXAMPLE 2.2. At a 4% annual interest rate, find the present value on January 1, 1988 of the following cash flow.

$1000 received on January 1, 1988
$2000 paid on January 1, 1989
$500 received on January 1, 1990
$1500 paid on January 1, 1991
$3000 received on January 1, 1992

Solution: The present value is

$$\begin{aligned} PV &= 1000 - 2000\,\frac{1}{1.04} + 500\left(\frac{1}{1.04}\right)^2 - 1500\left(\frac{1}{1.04}\right)^3 + 3000\left(\frac{1}{1.04}\right)^4 \\ &= \$770.12. \end{aligned}$$

So far, we have assumed in this chapter that the interest rate is given and we want to find the present value of a cash flow. However, we may also be interested in the opposite question. Let us see an example.

Suppose we have the opportunity to invest $1000 on January 1, 1990. In return, we will get $500 on January 1, 1991 and $600 on January 1, 1992. Should we make this investment or not? Of course, making this investment is better than leaving the money in the safe where it will remain unchanged. On the other hand, if the other option is to make a deposit at a bank at an annual rate of interest i, the money will also grow there. The deposit is more profitable than the investment if a deposit of $1000 on January 1, 1990 is more than what is needed to withdraw $500 on January 1, 1991 and $600 on January 1, 1992 from the account, that is, the sum of the present values of $500 and $600 is less than $1000. We can write this as

$$1000 > \frac{500}{1+i} + \frac{600}{(1+i)^2}\ .$$

Since the interest rate i is not fixed in this problem, we use the notation $PV(i)$ for the present value at interest rate i. So we get the inequality

$$PV(i) = -1000 + \frac{500}{1+i} + \frac{600}{(1+i)^2} < 0.$$

On the other hand, we will choose the investment if

$$PV(i) = -1000 + \frac{500}{1+i} + \frac{600}{(1+i)^2} > 0.$$

Now, $PV(i)$ can be interpreted as the present value on January 1, 1990 of the cash flow \$-1000, \$500, and \$600. So if $PV(i)$ is greater than zero, we choose the investment, otherwise we do not.

Let us see the present value of the cash flow at different interest rates

i	$PV(i)$
0.03	50.99
0.04	35.50
0.05	20.41
0.06	5.70
0.07	-8.65
0.08	-22.63

This implies that at $i = 3\%$ or $i = 6\%$ we choose the investment, but at $i = 7\%$ or $i = 8\%$ we do not. It seems that there must be an $i = i_0$ such that

$$PV(i) = 0,$$

$$PV(i) > 0, \text{ if } i < i_0,$$

and

$$PV(i) < 0 \text{ if } i > i_0.$$

This is clearly the case, as the following considerations show.

The function $PV(i)$ is continuous and strictly monotone decreasing in i for $i \geq 0$. If $i = 0$, then $PV(i) = 100$ is positive and the limit of $PV(i)$ at infinity is -1000; that is, negative. It is known from elementary calculus that there is exactly one positive i, say i_0, for which $PV(i) = 0$. Moreover, $PV(i) > 0$ for $i < i_0$ and $PV(i) < 0$ for $i > i_0$.

The interest rate i_0 which makes $PV(i)$ equal to zero is called the yield of the transaction.

In this example, i_0 can be obtained explicitly. From $PV(i) = 0$, we get

$$-1000(1+i)^2 + 500(1+i) + 600 = 0.$$

Using the quadratic formula, we obtain

$$i = 0.06394 \text{ and } i = -1.56394.$$

Since we are looking for a positive interest rate, the yield is 6.394%.

In general, assume sums of $C_1, C_2, \ldots, C_n$ are due at times $t_1, t_2, \ldots, t_n$, where $t_1 \leq t_2 \leq \ldots \leq t_n$. Choose a t_0 such that $t_0 \leq t_1$. Assume that (2) of Section 1.1 holds true, thus $v(t_0, t_i) = (1 + i)^{t_0 - t_i}$. Then the present value of the cash flow can be expressed as

$$PV_{t_0}(i) = C_1(1 + i)^{t_0 - t_1} + C_2(1 + i)^{t_0 - t_2} + \ldots + C_n(1 + i)^{t_0 - t_n}. \tag{3}$$

If there is an $i > -1$ which makes $PV_{t_0}(i)$ equal to zero, it is called the yield or internal rate of return of the transaction.

Note that $PV_{t_0}(i) = 0$ is equivalent to

$$C_1(1 + i)^{-t_1} + C_2(1 + i)^{-t_2} + \ldots + C_n(1 + i)^{-t_n} = 0. \tag{4}$$

Therefore, we do not need t_0 to define the yield. We can define it as a root $i > -1$ of (4). We may wonder whether it is possible for the yield of a transaction to be negative. It is possible as the following simple example shows.

We invest \$1000 for one year. However, the investment performs very poorly and we only get \$900 back after one year. The yield of this transaction is the root of

$$-1000 + 900(1 + i)^{-1} = 0,$$

from which

$$i = -0.1.$$

On the other hand, a yield less than or equal to -1 is not possible since this would imply that \$1 accumulates to a negative number in one year $(1 + i < 0)$. This would mean, we loose more than what we have invested, which is impossible.

The equation (4) can have no roots, one root, or more than one root. The general analysis of this equation is beyond the scope of this book. However, we will prove a useful theorem which gives a sufficient condition for the existence and uniqueness of the root of (4).

In order to state the theorem, we need to introduce the concept of net cash flows. Recall that in (3) and so in (4) it is possible that $t_i = t_{i+1}$. If we add all the payments due at the same time t, we get a net cash flow. That is, $C_1, C_2, \ldots, C_n$ is a net cash flow, if $t_1 < t_2 < \ldots < t_n$. Note also that when valuing a cash flow, $C_i = 0$ can be dropped.

THEOREM 2.1. *Let $C_1, C_2, \ldots, C_n$ be nonzero sums due at times $t_1, t_2, \ldots, t_n$, where $t_1 < t_2 < \ldots < t_n$. If there exists k $(1 \le k < n)$ such that $C_1, C_2, \ldots, C_k$ are positive and $C_{k+1}, C_{k+2}, \ldots, C_n$ are negative, or vice versa, the equation*

$$C_1(1+i)^{-t_1} + C_2(1+i)^{-t_2} + \ldots + C_n(1+i)^{-t_n} = 0 \tag{5}$$

has exactly one root i which satisfies $i > -1$. Moreover, if C_1 and $\sum_{j=1}^{n} C_j$ have different signs, the root of (5) is positive.

Proof: Without loss of generality, we can assume that $C_1, C_2, \ldots, C_k$ are positive and $C_{k+1}, C_{k+2}, \ldots, C_n$ are negative.

Let t^* be any number between t_k and t_{k+1}. Multiplying (5) by $(1+i)^{t^*}$, we get

$$\begin{aligned} C_1(1+i)^{t^*-t_1} + C_2(1+i)^{t^*-t_2} + \ldots + C_k(1+i)^{t^*-t_k} + C_{k+1}(1+i)^{t^*-t_{k+1}} \\ + \ldots + C_n(1+i)^{t^*-t_n} = 0. \end{aligned} \tag{6}$$

Let us denote the left hand side of (6) by $f(i)$.

Let us examine the terms of the sum separately.

First take a j such that $j \le k$. Then $t^* - t_j > t_k - t_j \ge 0$, and so $(1+i)^{t^*-t_j}$ is strictly monotone increasing in i. Since $C_j > 0$ also holds we find that $C_j(1+i)^{t^*-t_j}$ is strictly monotone increasing in i.

Next let $j > k$. Then $t^* - t_j < t_{k+1} - t_j \le 0$, and so $(1+i)^{t^*-t_j}$ is strictly monotone decreasing in i. Since $C_j < 0$ in this case, we see that $C_j(1+i)^{t^*-t_j}$ is strictly monotone increasing in i.

Thus the function $f(i)$ is strictly monotone increasing in i. When i approaches -1 from the right, the limit of $(1+i)^{t^*-t_j}$ is 0 for $t^* - t_j > 0$, and plus infinity for $t^* - t_j < 0$. Hence, the limit of $f(i)$ at -1^+ is minus infinity.

On the other hand, if i tends to infinity, the limit of $(1+i)^{t^*-t_j}$ is plus infinity for $t^* - t_j > 0$, and zero for $t^* - t_j < 0$. Consequently, the limit of $f(i)$ at infinity is plus infinity.

Thus, $f(i)$ increases strictly from minus infinity at -1^+ to plus infinity at infinity. Also, the function $f(i)$ is continuous for $i > -1$. We know from calculus that the equation $f(i) = 0$ has exactly one root for which $i > -1$.

Finally, consider the second statement of the theorem. We only have to prove that (5) has a positive root.

Let $C_1 > 0$ and $\sum_{j=1}^{n} C_j < 0$. By multiplying (5) by $(1+i)^{t_1}$, we obtain

$$C_1 + C_2(1+i)^{t_1-t_2} + \dots + C_n(1+i)^{t_1-t_n} = 0. \tag{7}$$

Let us denote the left hand side of (7) by $g(i)$. The function $g(i)$ is continuous for $i > -1$, $g(0) = \sum_{j=1}^{n} C_j < 0$, and the limit of $g(i)$ as i goes to infinity is $C_1 > 0$. Thus $g(i)$ has a positive root. ■

The usefulness of this theorem can be demonstrated by the following, very common transaction.

Assume we make an investment of I at time t_1. In return, we will receive payments of $R_2, R_3, \dots, R_n$ later on. Then, we can define a cash flow by $C_1 = -I$, $C_2 = R_2, C_3 = R_3, \dots, C_n = R_n$. Thus $C_1 < 0$ and $C_2, C_3, \dots, C_n$ are positive. Moreover, the transaction is only useful if we receive more than what we have invested. So we should have $I < R_1 + R_2 + \dots + R_n$; that is, $\sum_{j=1}^{n} C_j > 0$. Theorem 2.1 says that the yield equation has a unique solution, which is positive. Furthermore, let us consider the present value of this cash flow at t_1. Using (3), this is

$$PV_{t_1}(i) = C_1 + C_2(1+i)^{t_1-t_2} + \dots + C_n(1+i)^{t_1-t_n}.$$

Now we have $C_j > 0$ and $t_1 - t_j < 0$ for $j > 1$. Therefore, $PV_{t_1}(i)$ is strictly monotone decreasing in i.

EXAMPLE 2.3. Find the yield of the following transaction.

\$500 paid on January 1, 1988
\$1000 paid on January 1, 1989
\$400 received on January 1, 1990
\$1200 received on January 1, 1991

Should we make this transaction if our other option is to deposit the money at a 3% annual interest rate at a bank?

Solution: Since all the negative cash flows precede all the positive cash flows, it follows from Theorem 2.1 that the yield equation has a unique root. If we write down the present value of the cash flow on January 1, 1988, we get the equation

$$PV(i) = -500 - \frac{1000}{1+i} + \frac{400}{(1+i)^2} + \frac{1200}{(1+i)^3} = 0. \tag{8}$$

Let us compute the value of PV(i) for some i's.

i	$PV(i)$
0.02	34.8622
0.03	4.3346
0.04	-24.9203

If $i = 3\%$, the present value of the cash flow is positive, so we prefer the given transaction to the bank deposit. Since $PV(i)$ is positive for $i = 0.03$ and negative for $i = 0.04$, the root of (8) must be between these two numbers.

We use linear interpolation to obtain an approximation to the root of (8). We get

$$\begin{aligned} i &= 0.03 + \frac{0 - PV(0.03)}{PV(0.04) - PV(0.03)}\,(0.04 - 0.03) \\ &= 0.03 + \frac{-4.3346}{-24.9203 - 4.3346}\,0.01 = 0.031482. \end{aligned}$$

Hence, the yield is 3.148%.

Next, we turn our attention to continuous payment streams. Recall that in Section 1.1 we fixed t_0 and defined a nonnegative continuous function in t, $M(t_0,t)$, such that $M(t_0,t)$ gave the total payment made from t_0 to t. If $M(t_0,t)$ is differentiable in t, we called $\rho(t) = \frac{d}{dt} M(t_0,t)$ the rate of payment per annum at time t. Now we drop the condition that $M(t_0,t)$ be monotone increasing. Thus we can allow for negative payments, since $M(t_0,t_2) - M(t_0,t_1)$ is the payment made in the time interval from t_1 to t_2.

What is the present value of a continuous payment stream? Assume we are interested in the present value at t_0 of the payment stream $M(t_0,t)$, $t_0 \le t \le t_e$. Let us divide the interval (t_0,t_e) into n subintervals by the points $t_1 < t_2 < \dots < t_{n-1}$. Then, $M(t_0,t_{i+1}) - M(t_0,t_i)$ is the payment made between times t_i and t_{i+1}, $i = 0,1,\dots,n-1$. If t_i and t_{i+1} are close to each other, the present value at t_0 of the payment made in (t_i,t_{i+1}) can be approximated by

$$v(t_0,t_i)\,(M(t_0,t_{i+1}) - M(t_0,t_i)).$$

Hence an approximation of the present value of the cash flow at t_0 is

$$\begin{aligned} S_n &= \sum_{i=0}^{n-1} v(t_0,t_i)\,(M(t_0,t_{i+1}) - M(t_0,t_i)) \\ &= \sum_{i=0}^{n-1} v(t_0,t_i)\,\frac{M(t_0,t_{i+1}) - M(t_0,t_i)}{t_{i+1} - t_i}\,(t_{i+1} - t_i). \qquad (9) \end{aligned}$$

Now, assume that n goes to infinity, and the maximum of the differences $t_{i+1} - t_i$ $(i = 0,1,...,n - 1)$ tends to zero. Using elementary calculus, we can see easily that

$$\lim_{n\to\infty} S_n = \int_{t_0}^{t_e} v(t_0,t) \frac{d}{dt} M(t_0,t)dt.$$

Therefore, the present value at t_0 is

$$PV_{t_0} = \int_{t_0}^{t_e} v(t_0,t)\ \rho(t)dt. \tag{10}$$

Now, assume there is a continuous cash flow between t_0 and t_e at a rate $\rho(t)$ and we want to find its present value at a time preceding t_0, say at t_0^*. Let us define

$$\rho^*(t) = \begin{cases} 0 & \text{if } t_0^* \leq t < t_0 \\ \rho(t) & \text{if } t_0 \leq t \leq t_e \ . \end{cases}$$

Then $\rho^*(t)$ obviously defines the same cash flow as $\rho(t)$, and we can write

$$PV_{t_0^*} = \int_{t_0^*}^{t_e} v(t_0,t)\ \rho^*(t)dt = v(t_0^*,t_0) \int_{t_0}^{t_e} v(t_0,t)\ \rho(t)dt.$$

Thus,

$$PV_{t_0^*} = v(t_{t_0}^*,t_0)\ PV_{t_0}. \tag{11}$$

This is the same relationship as (2).

If the discount factor satisfies

$$v(t_0,t) = (1 + i)^{-(t-t_0)} = e^{-\delta(t-t_0)}, \tag{12}$$

then (10) can also be written as

$$PV_{t_0} = \int_{t_0}^{t_e} e^{-\delta(t-t_0)}\ \rho(t)dt. \tag{13}$$

As an important application of (13), consider the continuous payment stream at a constant rate r. Then we get

$$
\begin{aligned}
PV_{t_0} &= \int_{t_0}^{t_e} e^{-\delta(t-t_0)}\, r\, dt = r\, e^{\delta t_0} \int_{t_0}^{t_e} e^{-\delta t}\, dt \\
&= r\, e^{\delta t_0} \frac{e^{-\delta t_e} - e^{-\delta t_0}}{-\delta} \\
&= r\, \frac{1 - e^{-\delta(t_e-t_0)}}{\delta} \\
&= r\, \frac{1 - (1+i)^{-(t_e-t_0)}}{\delta} \,.
\end{aligned} \tag{14}
$$

EXAMPLE 2.4. A continuous payment is made from January 1, 1988 to January 1, 1991 at an annual rate of \$2000. Find its present value on January 1, 1988 and on January 1, 1986 using a 4% annual rate of interest.

Solution: Since $i = 0.04$, we have $\delta = \log 1.04 = 0.03922$. Thus the present value on January 1, 1988 is

$$PV = \$2000\, \frac{1 - (1.04)^{-3}}{0.03922} = \$5660.56.$$

Using (11) we obtain the present value on January 1, 1986

$$PV = (1.04)^{-2}\, \$5660.56 = \$5233.51.$$

The yield can be defined for a continuous cash flow as well. If there exists a δ which makes the present value (13) zero, the corresponding annual rate of interest $i = e^{\delta} - 1$ is called the yield of the transaction. In other words, i is the yield, if $\delta = \log(1 + i)$ satisfies

$$\int_{t_0}^{t_e} e^{-\delta(t-t_0)}\, \rho(t)dt = 0 \tag{15}$$

or equivalently,

$$\int_{t_0}^{t_e} e^{-\delta t}\, \rho(t)dt = 0.$$

It is also possible to combine discrete and continuous cash flows. Then the present value is

$$PV = C_1\, v(t_0,t_1) + C_2\, v(t_0,t_2) + \dots + C_n\, v(t_0,t_n) + \int_{t_0}^{t_e} v(t_0,t)\, \rho(t)dt. \quad (16)$$

If the discount factor satisfies condition (12), then (16) can be written as

$$PV = C_1\, e^{-\delta(t_1-t_0)} + C_2\, e^{-\delta(t_2-t_0)} + \dots + C_n\, e^{-\delta(t_n-t_0)} + \int_{t_0}^{t_e} e^{-\delta(t-t_0)}\, \rho(t)dt. \quad (17)$$

If the interest rate is not known, and there exists a δ which makes (17) equal to zero, then $i = e^{\delta} - 1$ is the yield of this transaction.

We have already talked about the accumulated value in Section 1.1. There we only computed accumulations of positive sums. However, as the considerations on present value showed, it is sensible to value negative sums as well.

Now, we state some results on accumulated values. Since they can be proved similarly to the results on present values, their proofs are omitted.

Assume sums of $C_1,C_2,\dots,C_n$ are due at times $t_1,t_2,\dots,t_n$, respectively, where $t_1 \le t_2 \le \dots \le t_n$. The amounts C_i can be negative. Then the accumulated value of this discrete cash flow at t_e, where $t_e \ge t_n$, is

$$\begin{aligned} AV_{t_e} &= C_1A(t_1,t_e) + C_2A(t_2,t_e) + \dots + C_nA(t_n,t_e) \\ &= \frac{C_1}{v(t_1,t_e)} + \frac{C_2}{v(t_2,t_e)} + \dots + \frac{C_n}{v(t_n,t_e)}\,. \end{aligned} \quad (18)$$

Assume we have a continuous payment stream $M(t_0,t)$, $t_0 \le t \le t_e$, whose rate of payment is $\rho(t)$. Then the accumulated value of this continuous cash flow at t_e is

$$AV_{t_e} = \int_{t_0}^{t_e} A(t,t_e)\, \rho(t)dt = \int_{t_0}^{t_e} \frac{\rho(t)}{v(t,t_e)}\, dt. \quad (19)$$

If (12) is satisfied, (19) can also be expressed as

$$AV_{t_e} = \int_{t_0}^{t_e} \mathrm{e}^{\delta(t_e-t)}\, \rho(t)dt. \quad (20)$$

Furthermore, if $\rho(t) = r$ then

$$AV_{t_e} = r\,\frac{(1+i)^{t_e-t_0} - 1}{\delta}\,. \quad (21)$$

If a cash flow contains both discrete and continuous elements, its accumulated value is the sum of the accumulated values of the discrete and the continuous parts.

If we take $t_e^* > t_e$, then

$$AV_{t_e^*} = AV_{t_e} A(t_e, t_e^*) = \frac{AV_{t_e}}{v(t_e, t_e^*)} . \tag{22}$$

Moreover, we can find a simple relationship between present values and accumulated values. Since we are working with compound interest, (1) can be expressed as

$$PV_{t_0} = C_1 \frac{v(t_0,t_e)}{v(t_1,t_e)} + C_2 \frac{v(t_0,t_e)}{v(t_2,t_e)} + \dots + C_n \frac{v(t_0,t_e)}{v(t_n,t_e)} .$$

Then, using (18) we get

$$PV_{t_0} = v(t_0,t_e)\, AV_{t_e}.$$

Similarly, for continuous cash flows, (10) and (19) imply that

$$PV_{t_0} = \int_{t_0}^{t_e} \frac{v(t_0, t_e)}{v(t,t_e)} \rho(t)dt = v(t_0,t_e)\, AV_{t_e}.$$

Thus,

$$PV_{t_0} = v(t_0,t_e)\, AV_{t_e} \tag{23}$$

is always true. This can also be expressed as

$$AV_{t_e} = A(t_0,t_e)\, PV_{t_0}. \tag{24}$$

EXAMPLE 2.5. Consider the cash flow of Example 2.2. Find its accumulated value on January 1, 1993 at a 4% annual interest rate.

Solution: The accumulated value is

$$\begin{aligned} AV &= 1000(1.04)^5 - 2000(1.04)^4 + 500(1.04)^3 - 1500(1.04)^2 + 3000(1.04) \\ &= \$936.97. \end{aligned}$$

We can also obtain the result using the relationship (24) between present value and accumulated value. In Example 2.2 we found that the present value on January 1, 1988 is

$$PV = \$770.12.$$

Hence, the accumulated value on January 1, 1993 is

$$\$770.12(1.04)^5 = \$936.97.$$

EXAMPLE 2.6. Consider the continuous cash flow of Example 2.4. Find its accumulated value on January 1, 1991 and on January 1, 1993 at a 4% annual rate of interest.

Solution: First, let us consider the accumulated value on January 1, 1991. From (21), we get

$$AV = \$2000 \, \frac{(1.04)^3 - 1}{0.03922} = \$6367.36.$$

Using (24) and the result of Example 2.4 we can also write

$$AV = \$5660.51(1.04)^3 = \$6367.30.$$

The difference between the two results is due to round-off errors. Using (22) we obtain the accumulated value on January 1, 1993

$$AV = \$6367.36(1.04)^2 = \$6886.94.$$

Next, we focus on a cash flow which is concentrated on the time period from t_0 to t_e. That means, the cash flow is zero outside this interval. Also, assume that the present value of the cash flow is zero at t_0. Then, it follows from (24) that the accumulated value at t_e is zero as well.

Now, let us select a time t between t_0 and t_e. Then one part of the cash flow takes place before t, and another part occurs after t. If there is also a transaction exactly at t, we assign it to one of the two parts. Let us denote the cash flow before t by CF_1 (past cash flow) and the cash flow after t by CF_2 (future cash flow). Since the present value of the total cash flow is 0 at t_e, we get

$$PV_{t_0}(CF_1) + PV_{t_0}(CF_2) = 0.$$

Thus,

$$A(t_0,t)\, PV_{t_0}(CF_1) + A(t_0,t)\, PV_{t_0}(CF_2) = 0$$

and using (2) and (24) we obtain

$$AV_t(CF_1) + PV_t(CF_2) = 0.$$

Therefore, the present value of CF_2 at t is the negative of the accumulated value of CF_1 at t.

Why are we interested in valuing a cash flow at a time t between t_0 and t_e? To see this, consider the following example.

A bank promises an investor to pay him \$100 on January 1, 1989, \$200 on January 1, 1990, \$400 on January 1, 1991, and \$300 on January 1, 1992. The bank uses a 3% annual rate of interest. Then the investor has to pay

$$I = \frac{100}{1.03} + \frac{200}{(1.03)^2} + \frac{400}{(1.03)^3} + \frac{300}{(1.03)^4} = \$918.21$$

to the bank on January 1, 1988. So the payments made and received by the investor form a cash flow whose present value is zero.

Assume the bank checks its books on January 1, 1991 before the first payments of the year are made. Then it is necessary to find out how much money is credited to the investor at that moment. In other words, how much money should be reserved for the investor if the future paying liabilities have to be met. This can be computed in two different ways. One possibility is to find the accumulated value of the \$918.21 investment minus the \$100 and \$200 payments. This is

$$918.27(1.03)^3 - 100(1.03)^2 - 200(1.03) = \$691.26.$$

Another possibility is to find the present value of the future payments of \$400 and \$300:

$$400 + \frac{300}{1.03} = \$691.26.$$

We can see that the two approaches give the same results. The reason that the two results have the same sign is that in the computation of the present value, the sums \$400 and \$300 had a positive sign and in the computation of the accumulated value, we changed the signs of \$400 and \$300 to negative, and gave \$918.21 a positive sign.

Let us analyze this example a little further.

The original investment is \$918.21 but the investor receives a total of \$1000 from the bank. We may ask when the interest of \$81.79 is paid.

The bank received a capital of \$918.21 from the investor on January 1, 1988. The interest on this amount is \$918.21(0.03) = \$27.55 for the first year. Thus when the bank pays \$100 on January 1, 1989, \$27.55 from this can be regarded as the interest payment. So the capital repayment in this payment is \$100 - \$27.55 = \$72.45. The outstanding capital is \$918.21 - \$72.45 = \$845.76. Note that this can also be expressed as \$918.21(1.03) - \$100, which is the accumulated value of the past cash flow. This is also equal to the present value of the future cash flow:

$$\frac{200}{1.03}+\frac{400}{(1.03)^2}+\frac{300}{(1.03)^3}=\$845.76.$$

The interest paid next year is $845.76(0.03) = $25.37. So the capital repayment part in the payment on January 1, 1990 is $200 - $25.37 = $174.63 and the outstanding capital is $845.76 - $176.63 = $671.13.

The interest in the payment on January 1, 1991 is $671.13(0.03) = $20.13. Thus the capital repayment is $400 - $20.13 = $379.87 and the outstanding capital is $671.13 - $379.87 = $291.26.

Finally, the payment on January 1, 1992 contains an interest $291.26(0.03) = $8.73 and a capital repayment of $300 - $8.73 = $291.27. Since this is the last payment, the outstanding capital should be zero. Indeed, $291.26 - $291.27 = 0. (The slight difference is due to round-off errors.)

We can summarize the payments in the following payment schedule.

Payment	Interest Content of Payment	Capital Repaid	Outstanding Capital After Payment
January 1, 1989	27.55	72.45	845.76
January 1, 1990	25.37	174.63	671.13
January 1, 1991	20.13	379.87	291.26
January 1, 1992	8.73	291.27	0
Total	81.78	918.22	

We can see that the sum of the interest contents of the payments give $81.79, and the sum of the capital repayments is $918.21. (The slight differences are again due to round-off errors.)

If we are only interested in the capital repayment and the interest content of one payment we do not need to fill in a whole table. First we need to find the outstanding capital just after the previous payment. This can be obtained either as the accumulated value of the past cash flow or as the present value of the future cash flow. The interest on this outstanding capital will be the interest content of the next payment we are focusing on, and the difference between the actual payment and the interest payment is the capital repayment.

EXAMPLE 2.7. How much has to be invested on January 1, 1989 if the investment provides payments of $400 on January 1, 1990, $200 on January 1, 1991, $300 on January 1, 1992, and $100 on January 1, 1993. A 4% annual rate of interest is used.

What are the interest contents and the capital repayment parts of the payments on January 1, 1991 and on January 1, 1993?

Solution: The investment is

$$I = \frac{400}{1.04} + \frac{200}{(1.04)^2} + \frac{300}{(1.04)^3} + \frac{100}{(1.04)^4} = \$921.71.$$

Now, look at the payment on January 1, 1991. We have to find the outstanding capital just after the previous payment; that is, on January 1, 1990.

Since the past cash flow only contains two terms, the original investment and the first payment and the future cash flow consists of three payments, we may prefer to determine the outstanding capital as the accumulation of past cash flow

$$AV = 921.71(1.04) - 400 = \$558.58.$$

The interest earned on this amount by January 1, 1990 is \$558.58(0.04) = \$22.34. Therefore, the interest content of the payment on January 1, 1991 is \$22.34 and the capital repayment is \$200 - \$22.34 = \$177.66.

Next, consider the payment on January 1, 1993. First, we determine the outstanding capital on January 1, 1992, just after the payment made on that day.

We calculate the outstanding capital as the present value of the future cash flow, since this cash flow only contains one term.

$$PV = \frac{100}{1.04} = \$96.15.$$

The interest earned on this amount by January 1, 1993 is \$96.15(0.04) = \$3.85. Thus the interest content of the last payment is \$3.85 and the capital repayment is \$100 - \$3.85 = \$96.15.

For the rest of the book, we assume that the interest satisfies $A(t_1,t_2) = (1+i)^{t_2-t_1}$.

PROBLEMS

2.1. An account earns 6% interest per annum. How much has to be deposited on January 1, 1988 if we want to make the following withdrawals: \$500 on January 1, 1989, \$800 on January 1, 1990, \$1200 on January 1, 1991, and \$1000 on January 1, 1993.

2.2. Consider the following cash flow:

\$800 paid on January 1, 1989,
\$1000 paid on January 1, 1990,
\$700 received on January 1, 1990,
\$500 paid on January 1, 1992,

$2000 received on January 1, 1993.

Based on a 5% annual interest rate, find the present value of the cash flow on

a) January 1, 1988,
b) January 1, 1989.

2.3. Obtain the yield of the following transaction

$1500 paid on January 1, 1989
$600 received on January 1, 1991
$600 received on January 1, 1992
$500 received on January 1, 1993.

Is this transaction prefereable to depositing the money at a 4% annual rate of interest at a bank?

2.4. A continuous payment is made from January 1, 1990 to January 1, 1993 at an annual rate of $1500. Find its present value on January 1, 1990 and January 1, 1988, using a 3% annual rate of interest.

2.5. Consider the following cash flow:

$700 paid on January 1, 1988,
$600 received on January 1, 1989,
$500 paid on January 1, 1991,
$800 received on January 1, 1992.

Based on a 3% annual rate of interest, find the accumulated value of the cash flow on

a) January 1, 1992,
b) January 1, 1994.

2.6. A continuous payment is made from January 1, 1989 to January 1, 1992 at an annual rate of $2000. Find its present value on January 1, 1989 and its accumulated value on January 1, 1992.

2.7. For an investment of $2332 on January 1, 1988, the investor receives $600 on January 1, 1989, $650 on January 1, 1990, $670 on January 1, 1991, and $720 on January 1, 1992.

a) Verify that the yield of the transaction is 5%.
b) Find the interest content and the capital repayment part of the payment on January 1, 1989.

c) Find the interest content and the capital repayment part of the payment on January 1, 1991.

2.8. A loan of \$8000 is taken out on January 1, 1987. The loan is to be repaid by four installments, calculated on the basis of an 8% annual interest rate. The first installment is \$2000 due on January 1, 1988, the second is \$2500 due on January 1, 1989, the third is \$2800 due on January 1, 1990.

a) Find the amount of the final installment, due on January 1, 1991.
b) Obtain the payment schedule.

1.3. ANNUITIES

Now, we want to focus on special types of cash flows, whose payments occur at regular time intervals (e.g. yearly, monthly, or daily). These cash flows are called annuities. Later on in the book, we will study annuities whose payments are contingent on the survival of the annuitant. Those annuities are called life annuities. If the payments of the annuity do not depend on the survival of a person, we are talking about an annuity-certain. They will be discussed in this section. If the payments are made in advance, that is, at the beginning of each time interval, the annuity is called an annuity-due. On the other hand, if the payments are made in arrears, that is, at the end of each time period, the annuity is called an annuity-immediate. These names are used traditionally, although they do not seem to be very logical. The first payment of an "annuity-immediate" is not made immediately at the beginning of the first payment period, rather, it is due at the end of it.

An annuity whose payments are equal is called a level annuity. We will study annuities whose payments are \$1, since any other level annuity can be obtained from this by a simple multiplication.

First we examine annuities that make payments once a year. They are called yearly annuities.

Let us consider an annuity that pays \$1 at the beginning of n consecutive years. This is an annuity-due.

The present value of this annuity-due at the beginning of the first year is denoted by $\ddot{a}_{\overline{n}|}$. The letter "a" stands for annuity, the symbol $\overline{n}|$ means that the payments are limited to n years. The two dots above "a" are used to distinguish the annuity-due from the annuity-immediate whose notation is $a_{\overline{n}|}$. If it is not clear from the context, what the annual rate of interest is, we can include it in the subscript: $\ddot{a}_{\overline{n}|i}$, $a_{\overline{n}|i}$.

Using the summation formula of the geometric sequence, we get

$$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + \ldots + v^{n-1} = \sum_{k=0}^{n-1} v^k = \frac{1 - v^n}{1 - v}. \qquad (1)$$

Since $1 - v = d,$ we have

$$\ddot{a}_{\overline{n}|} = \frac{1 - v^n}{d}, \qquad (2)$$

or equivalently

$$1 = d\,\ddot{a}_{\overline{n}|} + v^n. \qquad (3)$$

Formula (3) can also be obtained by general reasoning.

Assume $1 is invested at the beginning of the first year. The annual interest for the first year is paid in advance. That means d is paid at the beginning of the first year. At the beginning of the second year, the interest on $1 for the second year is paid and it is again $d,$ etc. Finally, an interest of d is paid at the beginning of year n. At the end of year $n,$ the outstanding capital is still $1. Now taking the present value at the beginning of the first year, we find that the originally invested $1 should equal the present value of the interest payments plus the present value of the $1 outstanding capital at the end of year n. The interest payments form an annuity-due, and their present value is $d\,\ddot{a}_{\overline{n}|}$. The present value of $1 remaining at the end of year n is v^n and we get

$$1 = d\,\ddot{a}_{\overline{n}|} + v^n.$$

EXAMPLE 3.1. An annuity of $500 per annum is payable for 20 years. The first payment occurs on January 1, 1991. What is the price of this annuity if it is bought on January 1, 1991? Use a 3% annual rate of interest.

Solution: We have to find the present value of the annuity on January 1, 1991. Since this is an annuity-due, the present value is

$$PV = 500\,\ddot{a}_{\overline{20}|}.$$

Since $i = 0.03,$ we have $d = \frac{0.03}{1.03} = 0.029126$ and from (2) we get

$$\ddot{a}_{\overline{20}|} = \frac{1 - \left(\frac{1}{1.03}\right)^{20}}{0.029126} = 15.3238.$$

Thus,

$$PV = \$7661.90.$$

EXAMPLE 3.2. A loan of $5000 is taken out on January 1, 1992. It has to be repaid by 15 equal installments payable yearly in advance. Based on an 8% annual rate of interest, determine the amount of the installments.

Solution: Denoting the annual installment by X, we get the equation

$$5000 = X\ddot{a}_{\overline{15}|}$$

from which it follows that

$$X = \frac{5000}{\ddot{a}_{\overline{15}|}}.$$

Now,

$$\ddot{a}_{\overline{15}|} = \frac{1 - \left(\frac{1}{1.08}\right)^{15}}{1 - \frac{1}{1.08}} = 9.2442,$$

and consequently, the annual payment is

$$X = \frac{5000}{9.2442} = \$540.88.$$

The accumulation of the annuity-due at the end of year n is denoted by $\ddot{s}_{\overline{n}|}$ or $\ddot{s}_{\overline{n}|i}$. Using (24) of Section 1.2, we get

$$\ddot{s}_{\overline{n}|} = (1+i)^n \ddot{a}_{\overline{n}|}, \tag{4}$$

so

$$\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d}. \tag{5}$$

EXAMPLE 3.3. An amount of $300 is deposited at a bank on January 1 of each year from 1981 to 1989. What is the accumulation on December 31, 1989? Use a 3% annual rate of interest.

Solution: The term of this annuity-due is 9 years (1989 - 1981 + 1). As a result, the accumulated value on December 31, 1989 is

$$AV = 300\,\ddot{s}_{\overline{9}|}.$$

Since $i = 0.03$, we have $d = \frac{0.03}{1.03} = 0.029126$ and using (5), we get

$$\ddot{s}_{\overline{9}|} = \frac{(1.03)^9 - 1}{0.029126} = 10.4639.$$

Thus

$$AV = \$3139.17.$$

Next, assume an annuity-due is purchased whose payments start in year $m + 1$ and continue until year $m + n$. So there are no payments made in the first m years.

This is called a deferred annuity-due. Its present value at the beginning of the first year is denoted by ${}_{m|}\ddot{a}_{\overline{n}|}$.

Since after the first m years, the payments of this annuity coincide with those of a non-deferred annuity, using (2) of Section 1.2, we obtain

$$_{m|}\ddot{a}_{\overline{n}|} = v^m\,\ddot{a}_{\overline{n}|} \tag{6}$$

and hence

$$_{m|}\ddot{a}_{\overline{n}|} = \frac{v^m - v^{m+n}}{d}\,. \tag{7}$$

There is also another way of evaluating a deferred annuity. Note that a series of payments of \$1 made in the years $m + 1, m + 2, \ldots, m + n$ can be regarded as the difference between two annuities. The first one pays \$1 in years $1, 2, \ldots, m + n$, the second one pays \$1 in years, $1, 2, \ldots, m$. Thus

$$_{m|}\ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{m+n}|} - \ddot{a}_{\overline{m}|}. \tag{8}$$

It is left to the reader to show algebraically that the right hand sides of (6) and (8) are equal.

Next, we look at annuities whose payments vary with time. We will examine two special types of varying annuities, one with linearly increasing payments and a second one whose payments form a geometric sequence.

Let us consider first an annuity with linearly increasing payments. More specifically, we focus on an increasing annuity-due that pays $\$k$ at the beginning of year k for every k from 1 to n.

The present value of this annuity at the beginning of the first year is denoted by $(I\ddot{a})_{\overline{n}|}$, and its accumulation at the end of year n is $(I\ddot{s})_{\overline{n}|}$.

That is, the annuity pays \$1 at the beginning of the first year, \$2 at the beginning of the second year, \$$n$ at the beginning of year n. This annuity can be expressed as the sum of n annuities. The first one is an n year annuity-due of \$1 per annum, the second one an n - 1 year annuity-due of \$1 per annum deferred for 1 year, the third one an n - 2 year annuity-due of \$1 per annum deferred for 2 years, etc. The last annuity is a 1 year annuity-due of \$1 per annum deferred for n - 1 year. Thus, we get

$$(I\ddot{a})_{\overline{n|}} = \ddot{a}_{\overline{n|}} + {}_1|\ddot{a}_{\overline{n-1|}} + {}_2|\ddot{a}_{\overline{n-2|}} + \dots + {}_{n-1}|\ddot{a}_{\overline{1|}} = \sum_{k=0}^{n-1} {}_k|\ddot{a}_{\overline{n-k|}}.$$

Using (7), we obtain

$$(I\ddot{a})_{\overline{n|}} = \sum_{k=0}^{n-1} \frac{v^k - v^n}{d} = \frac{\sum_{k=0}^{n-1} v^k - n\,v^n}{d},$$

and in view of (1), this can be written as

$$(I\ddot{a})_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - n\,v^n}{d}, \tag{9}$$

or equivalently

$$\ddot{a}_{\overline{n|}} = d\,(I\ddot{a})_{\overline{n|}} + n\,v^n. \tag{10}$$

Formula (10) can also be obtained by general reasoning.

Assume \$1 is invested at the beginning of each year in the n year long period. The interest on \$1 for the first year is paid in advance, at the beginning of the year. This interest is d. At the beginning of the second year, the total invetment is already \$2. So at the beginning of the second year $2d$ is the interest paid in advance for the second year, etc. Finally, $n \times d$ is the interest paid at the beginning of year n. By the end of year n, the capital has grown up to \$$n$. So the present value of the investments, $\ddot{a}_{\overline{n|}}$ should equal the present value of the interest payments $d(I\ddot{a})_{\overline{n|}}$, plus the present value of the \$$n$ capital at the end of year nv^n. Thus (10) follows immediately.

Multiplying both sides of (9) by $(1 + i)^n$, we obtain the accumulated value

$$(I\ddot{s})_{\overline{n|}} = \frac{\ddot{s}_{\overline{n|}} - n}{d}. \tag{11}$$

The increasing annuity introduced here makes it possible to value annuity-dues whose payments form any arithmetic sequence. That means, the payment is A in the first year, $A + B$ in the second year, ..., $A + (n - 1)B$ in year n. We can see that the present value of this annuity cannot be expressed directly from $(I\ddot{a})_{\overline{n|}}$ since the second payment does not equal two times the first payment. However, we may try to split this annuity into two others whose present values can be obtained easily. The first guess would be a level annuity of A plus an increasing annuity with payments $0, B, 2B, ..., (n - 1)B$. However, we again have the problem that the second payment does not equal two times the first payment. A better choice is to express the annuity as the sum of a level annuity of $A - B$ per annum and an increasing annuity with payments $B, 2B, ..., nB$. It can happen that $A - B$ becomes negative but it does not affect the computations.

So we can determine the present value of an annuity-due with payments $A, A + B, ..., A + (n - 1)B$ as follows:

$$PV = (A - B)\ddot{a}_{\overline{n|}} + B(I\ddot{a})_{\overline{n|}}. \tag{12}$$

EXAMPLE 3.4. The first payment of a yearly annuity is made on January 1, 1985 and is of amount \$2000. Each subsequent payment increases by \$300 yearly. The last payment is made on January 1, 1992. Determine the present value on January 1, 1985 and the accumulated value on December 31, 1992 at a 4% annual rate of interest.

Solution: We have $A = \$2000$ and $B = \$300$, thus using (12), the present value is

$$PV = (2000 - 300)\ddot{a}_{\overline{8|}} + 300(I\ddot{a})_{\overline{8|}}.$$

Since $i = 0.04$, we get $v = 0.96154$ and $d = 0.03846$. Using (2) we obtain

$$\ddot{a}_{\overline{8|}} = \frac{1 - (0.96154)^8}{0.03846} = 7.00209$$

and (9) gives

$$(I\ddot{a})_{\overline{8|}} = \frac{7.0021 - 8(0.96154)^8}{0.03846} = 30.07029.$$

Therefore, the present value is $1700(7.00209) + 300(30.07029) = \$20924.64$. The accumulated value is $(1.04)^8\ 20924.64 = \$28636.81$.

There are also annuities-due whose payments form a geometric sequence. That is, the payment is A in the first year, bA in the second year, ..., $b^{n-1}A$ in year n, where b is positive. Let us define j as

$$j = b - 1. \tag{13}$$

The number j can be negative but $j \geq -1$ is always true. The expression $100j$ gives the percentage change in annual payments.

Now, the present value of this annuity-due is

$$PV = A + vbA + v^2b^2A + \ldots + v^{n-1}b^{n-1}A.$$

Note that this is equivalent to valuing a level annuity-due of A per annum at an annual rate of interest i^*, where

$$\frac{1}{1 + i^*} = v^* = v \cdot b.$$

Using (13) we get

$$1 + i^* = \frac{1 + i}{1 + j},$$

so

$$i^* = \frac{i - j}{1 + j}. \tag{14}$$

As a result, the present value of the annuity-due with payments $A, bA, b^2A, \ldots, b^{n-1}A$ is

$$PV = A\,\ddot{a}_{\overline{n}|i^*} \tag{15}$$

where i^* is defined by (14).

Note that if i equals j then i^* is zero and so is d^*. Therefore, formula (2) cannot be used to compute $\ddot{a}_{\overline{n}|}$. However, in this case $v \cdot b = 1$, and $PV = A \cdot n$.

EXAMPLE 3.5. The payments of an annuity are made on January 1 of the years 1980 through 1990. The first payment is \$200 and the payments increase by 2% yearly. What is the price of this annuity on January 1, 1980, if a 3% annual rate of interest is used?

Solution: Using (15), the present value is $200\,\ddot{a}_{\overline{11}|i^*}$, where $i^* = \dfrac{0.03 - 0.02}{1.02} = 0.009804$. So $v^* = 0.990292$ and $d^* = 0.009708$. Thus,

$$\ddot{a}_{\overline{11}|0.009804} = \frac{1 - (0.990292)^{11}}{0.009708} = 10.4813$$

and the price of the annuity is 200(10.4813) = \$2096.26.

So far we have studied yearly annuities whose payments are made at the beginning of the years. If we consider annuities whose payments are of the same amount as those of the annuities-due, but the payments take place at the end of the years instead of at the beginning of them, we get the corresponding annuities-immediate. The only difference in notation is that we do not write the two dots above the letters a and s.

Note that a yearly annuity-immediate can be treated as a yearly annuity-due deferred for one year. So using (2) of Section 1.2, we get

$$PV(\text{annuity-immediate}) = v\, PV(\text{annuity-due}). \tag{16}$$

Moreover, taking (23) of Section 1.2 into account, we obtain

$$AV(\text{annuity-immediate}) = v\, AV(\text{annuity-due}). \tag{17}$$

It is also useful to keep in mind that since

$$d = 1 - v$$

we get

$$\frac{d}{v} = \frac{1}{v} - 1 = i,$$

that is

$$\frac{v}{d} = \frac{1}{i}. \tag{18}$$

Using the results for the annuity-due and applying (16), (17), and (18) we get

$$a_{\overline{n}|} = \frac{1 - v^n}{i}, \tag{19}$$

and

$$1 = i\, a_{\overline{n}|} + v^n. \tag{20}$$

The values of $a_{\overline{n}|}$ are tabulated in Appendix 1 for selected values of i and n.

Formula (20) can be obtained by general reasoning, which is very similar to the explanation given to (3). The only difference is that the interest payments are made at the end of each year. Therefore they are of amount i and form an annuity-immediate.

Furthermore, we obtain

$$s_{\overline{n}|} = (1+i)^n a_{\overline{n}|}, \tag{21}$$

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}, \tag{22}$$

$$_{m|}a_{\overline{n}|} = v^m a_{\overline{n}|}, \tag{23}$$

$$_{m|}a_{\overline{n}|} = \frac{v^m - v^{m+n}}{i}, \tag{24}$$

$$_{m|}a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|}, \tag{25}$$

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n\, v^n}{i}, \tag{26}$$

$$\ddot{a}_{\overline{n}|} = i\,(Ia)_{\overline{n}|} + n\, v^n, \tag{27}$$

and

$$(Is)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i}. \tag{28}$$

Note that on the right hand sides of (26), (27), and (28) the two dots are still above a and s. This also follows from general reasoning whose details are left to the reader.

Using (12) and (16) we get the following result.

The present value of an annuity-immediate with payments $A, A+B, \ldots, A+(n-1)B$ can be obtained as

$$PV = (A - B)a_{\overline{n}|} + B\,(Ia)_{\overline{n}|}. \tag{29}$$

On the other hand, the present value of an annuity with payments $A, bA, \ldots, b^{n-1}A$ cannot be obtained directly from (15). The present value is

$$PV = vA + v^2\, bA + v^3\, b^2 A + \ldots + v^n\, b^{n-1} A$$

$$= \frac{1}{b}(v\, bA + v^2 b^2 A + v^3 b^3 A + \ldots + v^n b^n A)$$

$$= \frac{1}{b} v^* \ddot{a}_{\overline{n}|i^*} A = \frac{A}{b} a_{\overline{n}|i^*}, \tag{30}$$

where i^* is defined by (14).

If $i = j$, we get

$$PV = \frac{A}{b}\, n.$$

EXAMPLE 3.6. An annuity of \$1200 per annum is payable for 15 years. The first payment is made on December 31, 1991. What is the price of the annuity if it is bought on January 1, 1991? Use a 4% annual rate of interest.

Solution: We have to find the present value of the annuity on January 1, 1991. Since this is an annuity-immediate, the present value is

$$PV = 1200\ a_{\overline{15}|}.$$

We have

$$a_{\overline{15}|} = \frac{1 - \left(\frac{1}{1.04}\right)^{15}}{0.04} = 11.1184.$$

Alternatively, we can use Appendix 1 to obtain $a_{\overline{15}|}$. It gives

$$a_{\overline{15}|} = 11.1184$$

as we have already computed it. Thus

$$PV = 1200(11.1184) = \$13342.06.$$

EXAMPLE 3.7. An amount of \$800 is deposited at a bank on December 31 of each year starting in 1980 and ending in 1992. What is the accumulation on December 31, 1992 if a 3% annual rate of interest is used?

Solution: We have to find the accumulation of a 13 year annuity-immediate. It can be expressed as

$$AV = 800\ s_{\overline{13}|}.$$

We can obtain $s_{\overline{13}|}$ as

$$s_{\overline{13}|} = \frac{(1.03)^{13} - 1}{0.03} = 15.6178$$

which is the same value as in Appendix 1. Hence

$$AV = 800(15.6178) = \$12494.24.$$

EXAMPLE 3.8. An amount of $500 is payable on December 31 for 15 years. The first payment is due in 1990. Using a 3% annual interest rate, find the purchase price of this annuity on January 1, 1980.

Solution: The purchase price is $\$500\ {}_{10|}a_{\overline{15}|}$. We can evaluate ${}_{10|}a_{\overline{15}|}$ in two ways. We can either write

$${}_{10|}a_{\overline{15}|} = v^{10} a_{\overline{15}|} = 0.74409(11.9379) = 8.8829$$

or we can compute it as

$${}_{10|}a_{\overline{15}|} = a_{\overline{25}|} - a_{\overline{10}|} = 17.4131 - 8.5302 = 8.8829.$$

Thus, the purchase price is $500(8.8829) = \$4441.45$.

EXAMPLE 3.9. An annuity pays $200 on December 31, 1980, $400 on December 31, 1981,...,$2200 on December 31, 1990. Find the present value of this annuity on January 1, 1980 and its accumulated value on December 31, 1990 at a 3% annual interest rate.

Solution: The present value is $\$200\ (Ia)_{\overline{11}|}$. Now

$$(Ia)_{\overline{11}|} = \frac{\ddot{a}_{\overline{11}|} - 11\,v^{11}}{i},$$

where

$$\ddot{a}_{\overline{11}|} = \frac{1 - \left(\frac{1}{1.03}\right)^{11}}{1 - \frac{1}{1.03}} = 9.5302$$

and

$$v^{11} = \left(\frac{1}{1.03}\right)^{11} = 0.72242.$$

Thus

$$(Ia)_{\overline{11}|} = \frac{9.5302 - 11(0.72242)}{0.03}$$
$$= 52.786.$$

Thus, the present value is $200(52.786) = \$10557.2$. The accumulated value is $(1.03)^{11}\ \$10557.2 = \$14613.6$.

The values of $(1 + i)^n$, v^n, $s_{\overline{n}|}$ and $a_{\overline{n}|}$ are tabulated in Appendix 1 for selected interest rates. Since $\ddot{s}_{\overline{n}|}$ and $\ddot{a}_{\overline{n}|}$ cannot be found in the tables, it is useful to find relationships between $\ddot{s}_{\overline{n}|}$ and $s_{\overline{n}|}$, and between $\ddot{a}_{\overline{n}|}$ and $a_{\overline{n}|}$. It follows from (16), (17), and (18) that

$$\ddot{a}_{\overline{n}|} = (1 + i)\, a_{\overline{n}|} = \frac{a_{\overline{n}|}}{v} = \frac{i}{d}\, a_{\overline{n}|} \tag{31}$$

and

$$\ddot{s}_{\overline{n}|} = (1 + i) s_{\overline{n}|} = \frac{s_{\overline{n}|}}{v} = \frac{i}{d}\, s_{\overline{n}|}. \tag{32}$$

In addition, note that if we drop the first payment from an annuity-due, we get an annuity-immediate with the number of payments decreased by 1, that is

$$\ddot{a}_{\overline{n}|} - 1 = a_{\overline{n-1}|},$$

and hence

$$\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}. \tag{33}$$

On the other hand, if we drop the last payment from an n year annuity-immediate, the resulting annuity can be regarded as an $n - 1$ year annuity-due deferred for 1 year. Therefore taking the accumulated value at the end of year n, we get

$$s_{\overline{n}|} - 1 = \ddot{s}_{\overline{n-1}|},$$

and so

$$\ddot{s}_{\overline{n}|} = s_{\overline{n+1}|} - 1. \tag{34}$$

Formulas (33) and (34) are easier to use than (31) and (32) since an addition or a subtraction is simpler than a multiplication or division.

Thus, in Example 3.1, we can calculate $\ddot{a}_{\overline{20}|}$ from Appendix 1 as

$$\ddot{a}_{\overline{20}|} = 1 + a_{\overline{19}|} = 1 + 14.3238 = 15.3238,$$

and in Example 3.3 we can write

$$\ddot{s}_{\overline{9}|} = s_{\overline{10}|} - 1 = 11.4639 - 1 = 10.4639.$$

Next we examine annuities whose annual sums are paid in more than one installment.

Consider any annuity-due discussed so far. Now assume that for every k from 1 to n, a payment of C_k at the beginning of the year is replaced by p payments of $\frac{C_k}{p}$ each. The first payment is made at the beginning of the year, the second payment $\frac{1}{p}$ years later, ..., the p^{th} payment $\frac{p-1}{p}$ years later. For a level annuity, we take $C_k = 1$, $k = 1,...,n$ and for a linearly increasing annuity, $C_k = k$, $k = 1,...,n$. We say that these annuities are payable $p$thly, and we indicate it by the symbol (p) in the superscript of a and s; for example $\ddot{a}^{(p)}_{\overline{n}|}$, $\ddot{s}^{(p)}_{\overline{n}|}$. How can we value these annuities?

Let us focus on the payments of only one year first. We can ask what the present value of this cash flow is at the beginning of the year. Note that the p payments of $\frac{C_k}{p}$ each form an annuity-due. The only difference compared to what we have discussed so far is that the time interval between two payments is only $\frac{1}{p}^{\text{th}}$ of a year. Hence if we use the effective rate of interest for a $\frac{1}{p}$ long period; that is, $\frac{i^{(p)}}{p}$, the present value can be obtained from (2) as

$$PV = \frac{C_k}{p} \frac{1 - \left(1 + \frac{i^{(p)}}{p}\right)^{-p}}{\frac{d^{(p)}}{p}}. \tag{35}$$

Thus, using (13) of Section 1.1 we obtain

$$PV = C_k \frac{1 - (1+i)^{-1}}{d^{(p)}} = C_k \frac{1 - v}{d^{(p)}} = C_k \frac{d}{d^{(p)}} . \tag{36}$$

Therefore, the present value of the p installment payments of $\frac{C_k}{p}$ during the year is the same as the present value of one payment of $C_k \frac{d}{d^{(p)}}$ at the beginning of the year.

So, if we replace the p installment payments of $\frac{C_k}{p}$ with one payment of $C_k \frac{d}{d^{(p)}}$ for all k's, the present value of the cash flow does not change. However, these newly created payments form a yearly annuity. We can use the formulas derived for yearly annuities and obtain

$$\ddot{a}^{(p)}_{\overline{n}|} = \frac{d}{d^{(p)}} \ddot{a}_{\overline{n}|} = \frac{1 - v^n}{d^{(p)}} ,$$

$$\ddot{s}^{(p)}_{\overline{n}|} = (1+i)^n \ddot{a}^{(p)}_{\overline{n}|} ,$$

$$\ddot{s}^{(p)}_{\overline{n}|} = \frac{d}{d^{(p)}} \ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d^{(p)}} ,$$

$${}_{m|}\ddot{a}^{(p)}_{\overline{n}|} = v^m \ddot{a}^{(p)}_{\overline{n}|} ,$$

$${}_{m|}\ddot{a}^{(p)}_{\overline{n}|} = \frac{d}{d^{(p)}} {}_{m|}\ddot{a}_{\overline{n}|} = \frac{v^m - v^{m+n}}{d^{(p)}} ,$$

$${}_{m|}\ddot{a}^{(p)}_{\overline{n}|} = \ddot{a}^{(p)}_{\overline{m+n}|} - \ddot{a}^{(p)}_{\overline{m}|} ,$$

$$(I\ddot{a})^{(p)}_{\overline{n}|} = \frac{d}{d^{(p)}} (I\ddot{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n\, v^n}{d^{(p)}} ,$$

and

$$(I\ddot{s})^{(p)}_{\overline{n}|} = \frac{d}{d^{(p)}} (I\ddot{s})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{d^{(p)}} .$$

Since the interest tables contain the values of $a_{\overline{n}|}$ and $s_{\overline{n}|}$, but not of $\ddot{a}_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$, it is useful to express $\ddot{a}^{(p)}_{\overline{n}|}$ and $\ddot{s}^{(p)}_{\overline{n}|}$ as

$$\ddot{a}^{(p)}_{\overline{n}|} = \frac{i}{d^{(p)}} a_{\overline{n}|},$$

and

$$\ddot{s}^{(p)}_{\overline{n}|} = \frac{i}{d^{(p)}} s_{\overline{n}|},$$

where we used (31) and (32).

The values of $\frac{i}{d^{(p)}}$ are given in Appendix 1 for selected values of i and p.

EXAMPLE 3.10. An annuity of \$600 per annum payable monthly in advance is purchased for a term of 6 years. Find its price if the annual rate of interest is 3%. Find its accumulated value at the end of year 6.

Solution: The price is $600\,\ddot{a}^{(12)}_{\overline{6}|}$. We can write

$$\ddot{a}^{(12)}_{\overline{6}|} = \frac{i}{d^{(12)}} a_{\overline{6}|}.$$

The value of $a_{\overline{6}|}$ can be determined by using (2) or from Appendix 1:

$$a_{\overline{6}|} = 5.4172.$$

We also have to find $\frac{i}{d^{(12)}}$ for $i = 0.03$. Since $d = \frac{0.03}{1.03} = 0.029126$, equation (37) of Section 1.1 gives $d^{(12)} = 12(1 - (1 - 0.029126)^{\frac{1}{12}}) = 0.029522$. Thus, $\frac{i}{d^{(12)}} = 1.01619$. From Appendix 1, we get $\frac{i}{d^{(12)}} = 1.016177$. The difference between the two numbers is due to round-off errors. Therefore, we have

$$\ddot{a}^{(12)}_{\overline{6}|} = 1.016177(5.4172) = 5.50483$$

and the price of the annuity is $600(5.50483) = \$3302.90$. The accumulated value at the end of year 6 is

$$\$3302.90 \cdot 1.03^6 = \$3943.84.$$

Let us turn our attention to annuities-immediate. Assume that for every k from 1 to n, a payment of C_k at the end of the year is replaced by p payments of $\frac{C_k}{p}$ each. The first payment is made at the end of the first $\frac{1}{p}$ year long period, the seond payment is made $\frac{1}{p}$ year later, ..., the p^{th} payment at the end of the year. Then we get annuities-immediate payable $p$thly and we use the symbol (p) again; for example, $a^{(p)}_{\overline{n}|}, s^{(p)}_{\overline{n}|}$.

Let us find the accumulated value of the installment payments during one year at the end of the year. Using (22), we get

$$AV = \frac{C_k}{p}\,\frac{\left(1 + \dfrac{i^{(p)}}{p}\right)^p - 1}{\dfrac{i^{(p)}}{p}} = C_k\frac{1 + i - 1}{i^{(p)}} = C_k\frac{i}{i^{(p)}}.$$

Then in view of (23) of Section 1.2 and (24) of Section 1.2 we find that the present value of the n-year long cash flow does not change if we replace the p installment payments of $\frac{C_k}{p}$ with one payment of $C_k\frac{i}{i^{(p)}}$ made at the end of the year k. This new cash flow is an annuity-immediate with yearly payments. So using the formulas for yearly annuities we obtain

$$a^{(p)}_{\overline{n}|} = \frac{i}{i^{(p)}}\, a_{\overline{n}|} = \frac{1 - v^n}{i^{(p)}},$$

$$s^{(p)}_{\overline{n}|} = (1 + i)^n\, a^{(p)}_{\overline{n}|},$$

$$s^{(p)}_{\overline{n}|} = \frac{i}{i^{(p)}}\, s_{\overline{n}|} = \frac{(1 + i)^n - 1}{i^{(p)}},$$

$$_{m|}a^{(p)}_{\overline{n}|} = v^m\, a^{(p)}_{\overline{n}|},$$

$$_{m|}a^{(p)}_{\overline{n|}} = \frac{i}{i^{(p)}}\, _{m|}a_{\overline{n|}} = \frac{v^m - v^{m+n}}{i^{(p)}},$$

$$_{m|}a^{(p)}_{\overline{n|}} = a^{(p)}_{\overline{m+n|}} - a^{(p)}_{\overline{m|}},$$

$$(Ia)^{(p)}_{\overline{n|}} = \frac{i}{i^{(p)}}(Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{i^{(p)}},$$

and

$$(Is)^{(p)}_{\overline{n|}} = \frac{i}{i^{(p)}}(Is)_{\overline{n|}} = \frac{\ddot{s}_{\overline{n|}} - n}{i^{(p)}}.$$

The values of $\frac{i}{i^{(p)}}$ are given in Appendix 1 for selected values of i and p.

EXAMPLE 3.11. An amount of \$200 is payable quarterly in arrears for 5 years. If the annual interest rate is 3%, what is the purchase price of the annuity?

Solution: The quarterly payments of \$200 can be thought of as the installments of an annuity of \$800 per annum payable quarterly in arrears. Therefore, the price is $800\, a^{(4)}_{\overline{5|}}$. We have

$$a^{(4)}_{\overline{5|}} = \frac{i}{i^{(4)}}\, a_{\overline{5|}}.$$

The values of $a_{\overline{5|}}$ and $\frac{i}{i^{(4)}}$ can be calculated directly or taken from Appendix 1:

$$a_{\overline{5|}} = 4.5797,$$

$$\frac{i}{i^{(4)}} = 1.011181.$$

Hence, $a^{(4)}_{5} = 1.011181(4.5797) = 4.63091$ and the price of the annuity is $800(4.63091) = \$3704.73$.

EXAMPLE 3.12. An amount of $200 is paid quarterly in arrears for 5 years. If the interest rate is 3% per annum convertible quarterly, what is the purchase price of the annuity?

Solution: A 3% interest rate convertible quarterly means that the effective rate of interest for a quarter is $\frac{0.03}{4} = 0.75\%$. Furthermore, there are $4 \times 5 = 20$ quarters is a 5 year long period. Therefore, the price of the annuity is $\$200\, a_{\overline{20|}0.0075}$. Now,

$$a_{\overline{20|}0.0075} = \frac{1 - \left(\frac{1}{1.0075}\right)^{20}}{0.0075} = 18.5080.$$

Thus the purchase price is $200(18.5080) = \$3701.60$.

Note that the last two examples look similar. The only difference is that while in Example 3.11, the 3% is the effective annual rate of interest, in Example 3.12 the effective annual rate of interest is $\left(1 + \frac{0.03}{4}\right)^4 - 1 =$ 0.03034. However, we do not have interest tables for $i = 3.034\%$, so we had to make the calculations by hand.

EXAMPLE 3.13. An annuity is payable monthly in arrears. It starts with a monthly payment of $200 in 1980, and the monthly payments increase by $10 each year. The last payment is made at the end of 1990. Find the present value of this annuity on January 1, 1980, and the accumulated value on December 31, 1990 at a 4% annual rate of interest.

Solution: Let us split this annuity into two parts. One is a level annuity paying $190 each month, the other one is an increasing annuity paying $10 monthly in 1980, $20 monthly in 1981,..., $110 monthly in 1990. The total annual payment of the level annuity is $12 \times 190 = \$2280$, and the total annual payments of the increasing annuity are $120 in 1980, $240 in 1981,...,$1320 in 1990. Thus, the present value is

$$PV = 2280\, a_{\overline{11|}}^{(12)} + 120\, (Ia)_{\overline{11|}}^{(12)}.$$

So, we have

$$PV = \frac{i}{i^{(12)}} \left(2280\, a_{\overline{11|}} + 120\, (Ia)_{\overline{11|}}\right).$$

From Appendix 1, we get

$$a_{\overline{11|}} = 8.7605$$

$$\ddot{a}_{\overline{11}|} = 1 + a_{\overline{10}|} = 1 + 8.1109 = 9.1109$$

$$\frac{i}{i^{(12)}} = 1.018204,$$

so

$$(Ia)_{\overline{11}|} = \frac{\ddot{a}_{\overline{11}|} - 11\left(\frac{1}{1.04}\right)^{11}}{0.04} = \frac{9.1109 - 11(0.64958)}{0.04} = 49.138.$$

Therefore, the present value is $1.108204[2280(8.7605) + 120(49.138)] =$ \$26341.45. The accumulated value is $\$26341.45(1.04)^{11} = \$40551.45$.

Next we discuss continuous annuities. In their notation a bar "–" is put above a and s. First consider a continuous payment stream at a rate of \$1 per annum for a term of n years. The present value of this annuity at the beginning of the first year can be obtained from (13) of Section 1.2

$$\bar{a}_{\overline{n}|} = \int_0^n e^{-\delta t}\, dt = \left.\frac{e^{-\delta t}}{-\delta}\right|_0^n = \frac{1 - e^{-\delta n}}{\delta} = \frac{1 - v^n}{\delta},$$

and the accumulated value of the annuity at the end of year n is

$$\bar{s}_{\overline{n}|} = (1 + i)^n\, \bar{a}_{\overline{n}|} = \frac{(1 + i)^n - 1}{\delta}.$$

The present value of a deferred continuous annuity can be expressed as

$$_{m|}\bar{a}_{\overline{n}|} = v^m\, \bar{a}_{\overline{n}|} = \frac{v^m - v^{m+n}}{\delta}$$

or as

$$_{m|}\bar{a}_{\overline{n}|} = \bar{a}_{\overline{m+n}|} - \bar{a}_{\overline{m}|}.$$

If the annuity is paid continuously at a rate of k per annum in year k ($k = 1,2,\ldots,n$), then we get

$$(I\bar{a})_{\overline{n}|} = \sum_{k=0}^{n-1} {}_{k|}\bar{a}_{\overline{n-k}|} = \sum_{k=0}^{n-1} \frac{v^k - v^n}{\delta} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{\delta}$$

and

$$(I\bar{s})_{\overline{n|}} = (1+i)^n (I\bar{a})_{\overline{n|}} = \frac{\ddot{s}_{\overline{n|}} - n}{\delta}.$$

The present values and accumulated values of annuities-immediate are tabulated in most interest tables. Therefore, it is useful to express the present values and accumulated values of continuous annuities in terms of those of annuities-immediate. We get

$$\bar{a}_{\overline{n|}} = \frac{i}{\delta} a_{\overline{n|}},$$

$$\bar{s}_{\overline{n|}} = \frac{i}{\delta} s_{\overline{n|}},$$

$$_{m|}\bar{a}_{n} = \frac{i}{\delta}\, _{m|}a_{\overline{n|}},$$

$$(I\bar{a})_{\overline{n|}} = \frac{i}{\delta} (Ia)_{\overline{n|}},$$

and

$$(I\bar{s})_{\overline{n|}} = \frac{i}{\delta} (Is)_{\overline{n|}}.$$

The values of $\frac{i}{\delta}$ are given in Appendix 1 for selected values of i.

Note that the expressions for the continuous annuities can be obtained as the limits of the $p$thly annuities as p goes to infinity taking into account that $\lim_{p\to\infty} i^{(p)} = \delta$. For example,

$$a^{(p)}_{\overline{n|}} = \frac{i}{i^{(p)}} a_{\overline{n|}},$$

thus

$$\bar{a}_{\overline{n|}} = \lim_{p\to\infty} \frac{i}{i^{(p)}} a_{\overline{n|}} = \frac{i}{\delta} a_{\overline{n|}}.$$

EXAMPLE 3.14. An annuity is paid continuously at a rate of $500 per annum for 7 years. What is the present value of the annuity at the beginning of the first year and what is the accumulation at the end of year seven? Use a 5% annual interest rate.

Solution: The present value of the annuity is $\$500\,\bar{a}_{\overline{7}|}$. We can obtain $\bar{a}_{\overline{7}|}$ as:

$$\bar{a}_{\overline{7}|} = \frac{i}{\delta}\,a_{\overline{7}|}.$$

The values of $\frac{i}{\delta}$ and $a_{\overline{7}|}$ can be computed directly or we can look them up in Appendix 1:

$$a_{\overline{7}|} = 5.7864,$$

$$\frac{i}{\delta} = 1.024797.$$

Thus

$$\bar{a}_{\overline{7}|} = 1.024797(5.7864) = 5.92989.$$

Therefore, the present value is $500(5.92989) = $2964.94. The accumulated value equals $\$500\,\bar{s}_{\overline{7}|}$. We can obtain $\bar{s}_{\overline{7}|}$ as

$$\bar{s}_{\overline{7}|} = \frac{i}{\delta}\,s_{\overline{7}|} = 1.024797(8.1420) = 8.34390.$$

Thus the accumulated value is $500(8.34390) = $4171.95.

We can also use the relationship between present value and accumulated value to find the accumulation:

$$\$2964.94(1.05)^7 = \$4171.97.$$

EXAMPLE 3.15. An annuity is payable continuously for 6 years. The rate of payment is $300 per annum in the first year, $600 per annum in the second year,..., $2400 in the eighth year. Find the present value and the accumulated value of the annuity at a 3% annual rate of interest.

Solution: The present value is \$300 $(I\bar{a})_{\overline{6}|}$. We have

$$(I\bar{a})_{\overline{6}|} = \frac{\ddot{a}_{\overline{6}|} - 6v^6}{\delta}.$$

From Appendix 1, we get

$$\ddot{a}_{\overline{6}|} = 1 + a_{\overline{5}|} = 1 + 4.5797 = 5.5797,$$

$$v^6 = 0.83748,$$

$$\delta = 0.029559,$$

and

$$(I\bar{a})_{\overline{6}|} = \frac{5.5797 - 6(0.83748)}{0.029559} = 18.7699\,.$$

Thus, the present value is $300(18.7699) = \$5630.97$ and the accumulation is $(1.03)^6\, 5630.97 = \$6723.67$.

Finally, let us study annuities whose payments are made at intervals of time length r years, where r is a positive integer, greater than 1.

Assume we have ℓ consecutive time intervals, each of length of r years. Let us denote the total length of these intervals by n $(n = \ell \cdot r)$. If regular payments are made at the beginning of each interval, we get an annuity-due, if the payments occur at the end of each interval, we have an annuity-immediate.

We could value these annuities using an effective rate of interest $i_{eff} = (1+i)^{\ell} - 1$. However, even if i is a "nice" number, e.g. 0.02, 0.03, or 0.05, $(1+i)^{\ell} - 1$ does not come out nice, e.g. $(1.02)^2 - 1 = 0.0404$, $(1.03)^4 - 1 = 0.1255$, or $(1.05)^{12} - 1 = 0.7959$, and tables are not available for these interest rates. In order to determine the present value of the annuity, we try to find a series of annual payments whose present value coincides with the present value of the original payments.

Let us assume \$1 is paid at the beginning of each r-year long period. Let us select one of these r-year long periods and place a payment of C at the beginning of each of the r years. These payments form an annuity-due whose present value at the beginning of the r-year long period is

$$PV = C\,\ddot{a}_{\overline{r}|}.$$

Setting this present value equal to $1, we get

$$C = \frac{1}{\ddot{a}_{\overline{r}|}}.$$

Repeating the same procedure for every r-year long period, we get an n-year annuity-due whose payments are of amount $\frac{1}{\ddot{a}_{\overline{r}|}}$ at the beginning of each year. Therefore, its present value at the beginning of the first year is

$$PV = \frac{\ddot{a}_{\overline{n}|}}{\ddot{a}_{\overline{r}|}}. \tag{37}$$

So this is the present value of the annuity-due with payments of $1 every r years. The accumulation of the annuity at the end of the last year is

$$AV = PV(1+i)^n = (1+i)^n \frac{\ddot{a}_{\overline{n}|}}{\ddot{a}_{\overline{r}|}} = \frac{\ddot{s}_{\overline{n}|}}{\ddot{a}_{\overline{r}|}}. \tag{38}$$

Next assume $1 is paid at the end of each r-year long period. We want to replace the payment at the end of an r-year long period with payments of D at the end of each of these r years, whose accumulated value at the end of the r-year long period equals $1. Since the r payments form an annuity-immediate, we get

$$D s_{\overline{n}|} = 1, \tag{39}$$

thus

$$D = \frac{1}{s_{\overline{n}|}}. \tag{40}$$

Making these substitutions in every r-year long period, the present value at the beginning of the first year does not change. So it is

$$PV = \frac{a_{\overline{n}|}}{s_{\overline{r}|}}. \tag{41}$$

Furthermore, the accumulation at the end of the last year is

$$AV = PV(1+i)^n = \frac{s_{\overline{n}|}}{s_{\overline{r}|}}. \tag{42}$$

EXAMPLE 3.16. An amount of $1000 is payable at the end of 6 consecutive three year long periods. What is the present value of this annuity at the beginning of the first period? What is the accumulated value at the end of the last period? Use a 3% annual rate of interest.

Solution: From (41), the present value is

$$PV = 1000\,\frac{1}{s_{\overline{3}|}}\,a_{\overline{18}|}.$$

From the table in Appendix 1, we get

$$s_{\overline{3}|} = 3.0909 \text{ and } a_{\overline{18}|} = 13.7535.$$

Thus the present value is $\$1000\,\frac{13.7535}{3.0909} = 4449.67$. Using (42), the accumulated value is

$$AV = 1000\,\frac{1}{s_{\overline{3}|}}\,s_{\overline{18}|}.$$

From the table in Appendix 1, we get

$$s_{\overline{18}|} = 23.4144.$$

Thus, the accumulated value is $\$1000\,\frac{23.4144}{3.0909} = \$7575.27$. Of course, we could also obtain the accumulated value directly from the present value: $\$4449.67(1.03)^{18} = \$7575.27$.

Note that we could find the same results by computing $1000\,a_{\overline{6}|i_e}$ and $1000\,s_{\overline{6}|i_e}$ at an effective rate of interest of $i_e = (1.03)^3 - 1 = 0.09273$. However, there is no table available for an interest rate of 9.273%. That is why we prefer applying (41) and (42).

At the end of Section 1.2, we analyzed the structure of the payments an investor receives for a capital invested. We found that each payment can be split into an interest payment part and a capital repayment part. At this point, the reader is advised to review that part of the book.

Here we want to study the situation where the payments form an annuity.

Assume an amount of C is invested at the beginning of year one, and in return for this, the investor receives equal payments of A at the end of year $1,2,\ldots,n$. If the investment earns interest at an annual rate of i, we can write

$$C = A\, a_{\overline{n}|}.$$

Thus, the amount of each payment is

$$A = \frac{C}{a_{\overline{n}|}}. \tag{43}$$

We want to find out what the outstanding capital is just after the k^{th} payment is made ($k = 1,2,\ldots,n$). This is equal to the present value of an annuity of A per annum payable for $n - k$ years; that is, $A\, a_{\overline{n-k}|}$. From (43) we get

$$A\, a_{\overline{n-k}|} = C\frac{a_{\overline{n-k}|}}{a_{\overline{n}|}}. \tag{44}$$

The interest on this amount for one year gives the interest content of the $(k+1)^{\text{th}}$ payment. This equals

$$C\,\frac{a_{\overline{n-k}|}}{a_{\overline{n}|}}\,i = C\frac{i\, a_{\overline{n-k}|}}{a_{\overline{n}|}}.$$

From (20), we get $i\, a_{\overline{n-k}|} = 1 - v^{n-k}$. Thus, the interest content of the $(k+1)^{\text{th}}$ payment is

$$C\frac{1 - v^{n-k}}{a_{\overline{n}|}} = A\,(1 - v^{n-k}). \tag{45}$$

Therefore, the capital repayment part of the $(k+1)$th payment is

$$\frac{C}{a_{\overline{n}|}} - C\frac{1 - v^{n-k}}{a_{\overline{n}|}} = C\frac{v^{n-k}}{a_{\overline{n}|}} = A\, v^{n-k}. \tag{46}$$

If we choose the special case of the investment with $C = a_{\overline{n}|}$ then all the above formulas become simpler. In fact, the annual payments will be just \$1, and the payment schedule can be laid out as follows.

It can be seen from this schedule that the interest content of the payments is decreasing, while the capital repaid is increasing with time.

Payment	Interest Content of Payment	Capital Repaid	Outstanding Capital after Payment
1	$1 - v^n$	v^n	$a_{\overline{n-1}\rceil}$
2	$1 - v^{n-1}$	v^{n-1}	$a_{\overline{n-2}\rceil}$
$\vdots$			
k	$1 - v^{n-k+1}$	v^{n-k+1}	$a_{\overline{n-k}\rceil}$
$\vdots$			
$n - 1$	$1 - v^2$	v^2	$a_{\overline{1}\rceil}$
n	$1 - v$	v	0
Total	$n - a_{\overline{n}\rceil}$	$a_{\overline{n}\rceil}$	

EXAMPLE 3.17. An investment of $5000 is made on January 1, 1970. In return for this, the investor receives annual payments of equal amounts at the end of each year from 1970 to 1990.

a) What is the amount of each payment?

b) What is the interest content and the capital repayment part of the payment at the end of 1972?

c) What is the interest content and the capital repayment part of the payment at the end of 1990?

d) After which payment does the outstanding capital first drop below $3000?

e) What is the first year when the capital repayment is higher than the interest content?

Assume the investment earns interest at a 5% annual rate.

Solution: The number of payments is $n = 21$.

a) If the annual payment is denoted by A, we have $5000 = A\, a_{\overline{21}\rceil}$. Thus,

$$A = \frac{5000}{a_{\overline{21}\rceil}} = \frac{5000}{12.8212} = 389.98.$$

So the amount of each payment is $389.98.

b) In the year 1972, the third payment takes place. Thus, from (45), the interest content is

$$\$389.98(1 - v^{21-2}) = \$389.98(1 - 0.39573) = \$235.65.$$

Hence, the capital repaid is 389.98 - 235.65 = \$154.33. The capital repaid could also be obtained from (46)

$$\$389.98\, v^{21-2} = \$389.98(0.39573) = \$154.33.$$

c) In the year 1990, the 21^{st} payment is made, and therefore the interest content is

$$\$389.98(1 - v^{21-20}) = \$389.98(1 - 0.95238) = \$18.57,$$

and the capital repaid is 389.98 - 18.57 = \$371.41.

d) The outstanding capital after the k^{th} payment is

$$\$389.98\, a_{\overline{21-k}|}.$$

We have to find the smallest k such that

$$\$389.98\, a_{\overline{21-k}|} < \$3000;$$

that is,

$$a_{\overline{21-k}|} < 7.6927.$$

Checking the table in Appendix 1, we see that $21 - k \leq 9$ must be true. Thus, $k \geq 12$. So the smallest k is 12, which corresponds to year 1981.

e) The capital repayment in year k is $A\, v^{n-(k-1)}$ and the interest payment is $A(1 - v^{n-(k-1)})$. Hence we need to find the smallest k for which

$$A\, v^{n-(k-1)} > A(1 - v^{n-(k-1)}).$$

Simplifying, we get

$$v^{n-k+1} > 1 - v^{n-k+1},$$

that is

$$v^{n-k+1} > \frac{1}{2}.$$

Taking logarithm on both sides of the inequality, we obtain

$$(n - k + 1)\log v > -\log 2.$$

Now, $\log v = -\delta = -0.048790$, and $\log 2 = 0.693147$, and $n = 21$, so we get

$$(22 - k)(-0.048790) > (-0.693147)$$

$$22 - k < 14.2067$$

$$k > 7.7933.$$

Therefore, the smallest k is 8 which corresponds to year 1977.

Note that k could also be obtained from Appendix 1. From that table, we find $v^{14} > \frac{1}{2}$, $v^{15} < \frac{1}{2}$, hence $22 - k \leq 14$, and this also gives $k = 8$.

PROBLEMS

3.1. Given $i = 6\%$, obtain

a) $\ddot{a}_{\overline{1}|}$
b) $\ddot{a}_{\overline{10}|}$
c) $\ddot{s}_{\overline{10}|}$
d) $\ddot{s}_{\overline{20}|}$.

3.2. An annuity pays \$500 yearly in advance for 20 years. Using a 4% annual rate of interest, find

a) the present value of the annuity at the beginning of the first year.
b) the accumulated value of the annuity at the end of year 20.

3.3. Given $i = 8\%$, determine

a) ${}_{5|}\ddot{a}_{\overline{15}|}$
b) ${}_{12|}\ddot{a}_{\overline{25}|}$.

3.4. A 10 year level annuity-due of \$700 per annum is purchased four years before the payment period starts. Find the purchase price of the annuity based on a 3% annual rate of interest.

3.5. A loan of \$8000 is repaid by 8 equal installments payable yearly in advance with the first payment due 2 years after the loan is taken out. Based on a 9% annual rate of interest, find the amount of the annual installments.

3.6. Given $i = 0.02$, find

a) $(I\ddot{a})_{\overline{12}|}$
b) $(I\ddot{a})_{\overline{18}|}$
c) $(I\ddot{s})_{\overline{18}|}$.

3.7. An annuity is payable yearly in advance for 5 years. The payment in the first year is \$1000 and the payments increase by \$200 each year. Find

a) the present value of the annuity at the beginning of the first year.
b) the accumulated value of the annuity at the end of year 5.

Use a 4% annual rate of interest.

3.8. An annuity is payable yearly in advance for 10 years. The first payment is \$500 and each subsequent payment decreases by 3%. Based on a 5% annual interest rate, find the present value of the annuity at the beginning of the first year.

3.9. Given $i = 4\%$, determine

a) $a_{\overline{1}|}$
b) $a_{\overline{14}|}$
c) $s_{\overline{14}|}$
d) $s_{\overline{25}|}$.

3.10. Find the present value of a 20 year level annuity-immediate of \$800 per annum at the beginning of the first year based on a 3% annual rate of interest. Also find the accumulated value at the end of the year 20.

3.11. A loan of \$9000 is repaid by a 10 year level annuity-immediate. Find the amount of the annual installment based on a 7% annual interest rate.

3.12. Given $i = 5\%$, determine

a) $_{20|}a_{\overline{10}|}$
b) $_{3|}a_{\overline{15}|}$.

3.13. The first payment of a 15 year level annuity of \$900 per annum is made five years after the annuity is purchased. Show that the purchase price (say X) can be obtained from either of the following two equations

$$X = 900\ {}_{5|}\ddot{a}_{\overline{15}|}$$
$$X = 900\ {}_{4|}a_{\overline{15}|}.$$

Evaluate the purchase price on the basis of a 5% annual rate of interest.

3.14. Given $i = 3\%$, determine

a) $(Ia)_{\overline{10}|}$
b) $(Ia)_{\overline{20}|}$
c) $(Is)_{\overline{25}|}$.

3.15. A loan is repaid by a 12 year varying annuity-immediate. The first installment is \$1080 and the installments decrease by \$90 each year. Determine the amount of the loan using a 5% annual rate of interest.

3.16. The payments of a 10 year annuity-immediate increase by 3% each year. What is the present value of the annuity at the beginning of the first year if the amount of the first payment is \$550 and a 2% annual rate of interest is used. Also obtain the accumulated value of the annuity at the end of the year 10.

3.17. Given $i = 6\%$, obtain

a) $a^{(2)}_{\overline{15}|}$
b) $s^{(12)}_{\overline{20}|}$
c) ${}_{5|}\ddot{a}^{(6)}_{\overline{22}|}$
d) $(Ia)^{(4)}_{\overline{8}|}$

e) $(I\ddot{s})^{(12)}_{\overline{16}|}$.

3.18. A 10 year annuity of \$900 per annum is payable monthly in advance. Based on a 4% annual rate of interest, determine the present value of the annuity at the beginning of the first year and the accumulated value at the end of year 10.

3.19. A loan of \$5000 is repaid by equal payments made quarterly in advance during a term of 6 years. Using a 7% annual interest rate, find the amount of the quarterly payments.

3.20. An annuity-due of \$1200 per annum is payable monthly for 20 years. Obtain the present value of the annuity at the beginning of the first year on the basis of a

a) 5% annual rate of interest
b) 5% annual rate of interest convertible monthly.

3.21. The monthly installments of a 15 year annuity-due are \$300 in the first year and increase by \$20 each year. Based on a 6% annual rate of interest, find the present value of the annuity at the beginning of the first year and the accumulated value at the end of the last year.

3.22. Given $i = 5\%$, find

a) $\bar{a}_{\overline{10}|}$

b) $\bar{s}_{\overline{20}|}$

c) $_{8|}\bar{a}_{\overline{7}|}$

d) $(I\bar{a})_{\overline{3}|}$.

3.23. An annuity of \$2000 per annum is paid continuously over a period of 10 years. Using a 4% annual rate of interest, determine the present value of the annuity at the beginning of the first year and the accumulated value at the end of year 10.

3.24. The payment of a continuous 15 year annuity of \$3000 per annum starts 3 years after the purchase time. Determine the price of the annuity on the basis of a 6% annual rate of interest.

3.25. An annuity of \$1500 per annum is payable for 10 years. Based on a 5% annual rate of interest, find the present value of the annuity at the beginning of the first year if the payments are made

a) yearly in advance.
b) yearly in arrears.
c) monthly in advance.
d) monthly in arrears.
e) continuously.

3.26. Prove

$$a_{\overline{n}|} < a^{(p)}_{\overline{n}|} < \bar{a}_{\overline{n}|} < \ddot{a}^{(p)}_{\overline{n}|} < \ddot{a}_{\overline{n}|} \qquad (p > 1)$$

algebraically and also by general reasoning.

3.27. An amount of \$400 is paid at the beginning of 5 two year long periods. Based on a 4% annual interest rate find the present value of the annuity at the beginning of the first year.

3.28. A loan of \$6000 is repaid in equal installments payable yearly in arrears for 10 years.

a) Determine the amount of the annual payment.
b) Find the interest content and the capital repayment part of the payment at the end of the first year.
c) Find the interest content and the capital repayment part of the payment at the end of year six.
d) After which payment does the outstanding capital first drop below \$5000?
e) Find the first year when the capital repayment is higher than the interest content.

Use a 6% annual interest rate.

CHAPTER 2

MORTALITY

Life insurance is concerned with financial transactions whose payments depend on death or survival of the policyholder. For example, a life annuity makes regular payments until the insured dies, while other insurances pay a fixed sum on death. On the other hand, if the premiums for a life insurance are paid over a longer period of time, then their payment is contingent on survival.

Therefore, in order to be able to discuss life insurance, we need to study the theory of mortality first.

2.1. SURVIVAL TIME

Since the age at which someone will die cannot be predicted with certainty, the lifetime of a person can best be modeled by a random variable. We will denote the future lifetime of a newborn baby by the random variable T. The random variable T is also called the survival time.

Recall from probability theory that $F(t)$ denotes the probability that T is less than or equal to t:

$$F(t) = P(T \leq t). \tag{1}$$

The function $F(t)$ is called the distribution function of T. It is reasonable to assume that the distribution function of the survival time T is "smooth"; that is, $F(t)$ is differentiable and its derivative, $\frac{d}{dt} F(t) = f(t)$ is continuous. Therefore, $F(t)$ can be written as

$$F(t) = \int_0^t f(s)ds. \tag{2}$$

The function $f(t)$ is called the probability density function of T.

It is useful to introduce the concept of survival function. The survival function is denoted by $S(t)$ and is defined by

$$S(t) = 1 - F(t); \tag{3}$$

that is,

$$S(t) = P(T > t). \tag{4}$$

Hence $S(t)$ is the probability that an individual survives longer than t. It follows from (3) that the survival function $S(t)$ defines the distribution of T uniquely.

Note that since T is a continuous random variable, (4) is equivalent to

$$S(t) = P(T \geq t). \tag{5}$$

Since

$$1 = \int_0^\infty f(s)ds \tag{6}$$

is always true and (2) holds, we can write

$$S(t) = \int_t^\infty f(s)ds. \tag{7}$$

It follows from (7) that

$$\frac{d}{dt} S(t) = -f(t). \tag{8}$$

In most cases, when a life insurance policy is issued, the insured is not a newborn baby any more. If the insured is of age t at the time of the purchase of the policy, we are interested in the behavior of the survival time under the condition the individual has lived to age t. Hence, conditional probabilities play an important role in actuarial mathematics. Let us study them next.

Let us look at the conditional probability $P(T \geq t_2 \mid T \geq t_1)$, where $S(t_1) > 0$. We can write

$$P(T \geq t_2 \mid T \geq t_1) = \frac{P(T \geq t_2)}{P(T \geq t_1)} = \frac{S(t_2)}{S(t_1)}. \tag{9}$$

So if we define the function in two variables

$$f_0(t_1,t_2) = P(T \geq t_2 \mid T \geq t_1) = \frac{S(t_2)}{S(t_1)} \tag{10}$$

then $f_0(t_1,t_2)$ satisfies $f_0(t_1,t_2)\, f_0(t_2,t_3) = f_0(t_1,t_3)$. Hence, using Theorem 1.1 of Section 1.1, we can find a continuous function $g(t)$, such that

$$f_0(t_1,t_2) = e^{\int_{t_1}^{t_2} g(t)dt} \quad \text{and} \tag{11}$$

$$g(t) = \frac{d}{dt} \log f_0(t_1,t). \tag{12}$$

Now, (12) is equivalent to

$$g(t) = \frac{1}{S(t)} \frac{d}{dt} S(t) = \frac{-f(t)}{S(t)}.$$

Since $f(t)$ is a probability density function, it is always nonnegative. This makes $g(t)$ a negative function. However, we prefer to work with positive functions, so we define a new function $h(t)$ by

$$h(t) = \frac{f(t)}{S(t)} = -\frac{1}{S(t)} \frac{d}{dt} S(t). \tag{13}$$

Then $h(t)$ is nonnegative, and we have

$$P(T \geq t_2 \mid T \geq t_1) = \frac{S(t_2)}{S(t_1)} = e^{-\int_{t_1}^{t_2} h(t)dt}. \tag{14}$$

Taking $t_1 = 0$ in (14), we have $S(t_1) = 1$, so

$$P(T \geq t) = S(t) = e^{-\int_0^t h(s)ds}. \tag{15}$$

It follows from (13) that

$$f(t) = h(t)\, S(t); \tag{16}$$

thus we can express $f(t)$ in terms of $h(t)$:

$$f(t) = h(t)\, e^{-\int_0^t h(s)ds}. \tag{17}$$

Therefore, $h(t)$ defines the distribution of T uniquely.

Let us see how we can interpret the function $h(t)$. Since

$$P(T < t_2 \mid T \geq t_1) + P(T \geq t_2 \mid T \geq t_1) = \frac{P(t_1 \leq T < t_2)}{P(T \geq t_1)} + \frac{P(T \geq t_2)}{P(T \geq t_1)} = \frac{P(T \geq t_1)}{P(T \geq t_1)} = 1, \tag{18}$$

using (14), we obtain

$$P(T \leq t_2 \mid T \geq t_1) = 1 - e^{-\int_{t_1}^{t_2} h(t)dt} . \tag{19}$$

If t_1 and t_2 are very close to each other, then an approximation of the right hand side of (19) gives

$$P(T \leq t_2 \mid T \geq t_1) \approx 1 - e^{-(t_2 - t_1)h(t_1)} \approx 1 - (1 - (t_2 - t_1)\, h(t_1)) = (t_2 - t_1)\, h(t_1). \tag{20}$$

That means assuming an individual has survived to age t, the probability that he/she will die in the following ε-long period, where ε is small, is approximately $\varepsilon\, h(t)$. We say that $h(t)$ gives the conditional failure rate of the survival time T. The function $h(t)$ is called the hazard function of T.

EXAMPLE 1.1. Assume the lifetime of an individual follows the exponential distribution with probability density function

$$f(t) = \lambda\, e^{-\lambda t} \text{ for } t > 0.$$

Find the survival function and the hazard function.

Solution: The survival function is

$$S(t) = P(T \geq t) = \int_t^{\infty} \lambda\, e^{-\lambda s}\, ds = \int_t^{\infty} \lambda\, e^{-\lambda(s-t)}\, e^{-\lambda t}\, ds = e^{-\lambda t} \int_0^{\infty} \lambda\, e^{-\lambda u}\, du = e^{-\lambda t}, \text{ if } t \geq 0$$

and the hazard function is

$$h(t) = \frac{f(t)}{S(t)} = \frac{\lambda\, e^{-\lambda t}}{e^{-\lambda t}} = \lambda, \text{ if } t \geq 0.$$

Hence the hazard function is a constant. It is left to the reader to show that the exponential distribution is the only continuous distribution with a constant hazard function.

What we have discussed in this section belongs to a special field of probability theory, known as the survival analysis.

PROBLEMS

1.1. Assuming the lifetime of a person is given by an exponential distribution with an expected value of 80 years, find the probability the person will survive to age 50. What is the value of the hazard function at this age?

1.2. Given the survival function

$$S(t) = e^{-(\lambda t)^{\gamma}}, \text{ if } t \geq 0$$

where $\lambda > 0$ and $\gamma > 0$, obtain the probability density function of the lifetime random variable and the hazard function.

1.3. Prove that if the hazard function of a continuous distribution is constant, the distribution is exponential.

2.2. ACTUARIAL FUNCTIONS OF MORTALITY

In this section, we introduce some standard actuarial notations and derive important results of mortality theory.

As we have seen in Section 2.1, the survival function $S(t)$ defines the distribution of T uniquely, so we can use the values of $S(t)$ to describe the distribution of T. If we want to tabulate the values of $S(t)$, we have to list a lot of small numbers; that is, numbers between 0 and 1. However, it is more convenient to work with numbers greater than 1.

Let us choose a "large" positive number ℓ_0. Usually, ℓ_0 is somewhere between 10,000 and 1,000,000, but other choices of ℓ_0 are also possible. Let us multiply $S(t)$ by ℓ_0 for all $t \geq 0$ and denote the result by ℓ_t:

$$\ell_t = \ell_0 \, S(T). \tag{1}$$

We can interpret (1) as follows. Assume we observe ℓ_0 persons who have just been born and their lifetimes are identically distributed. What is the expected number of persons from this group who survive to age t? Let us denote the survival time of person i by T_i, $i = 1,2,\ldots,\ell_0$. Using the indicator function I, whose definition is

$$I(\text{statement}) = \begin{cases} 1 & \text{if the statement is true} \\ 0 & \text{if the statement is false,} \end{cases}$$

the number of survivors at age t can be expressed as

$$\sum_{i=1}^{\ell_0} I(T_i \geq t),$$

and therefore the expected number of survivors at age t is

$$\begin{aligned} E\left(\sum_{i=1}^{\ell_0} I(T_i \geq t)\right) &= \sum_{i=1}^{\ell_0} E(I(T_i \geq t)) \\ &= \sum_{i=1}^{\ell_0} (1 \cdot P(T_i \geq t) + 0 \cdot P(T_i < t)) \\ &= \sum_{i=1}^{\ell_0} P(T_i \geq t) \\ &= \ell_0\, S(t). \end{aligned} \tag{2}$$

It follows from (1) and (2) that ℓ_t is the expected number of persons surviving to age t.

Of course, if the values of ℓ_t are known for all $t \geq 0$, they define the distribution of T uniquely since (1) implies that

$$S(t) = \frac{\ell_t}{\ell_0}. \tag{3}$$

In actuarial mathematics, we use the function ℓ_t instead of $S(t)$. Moreover, the age is usually denoted by x instead of t. If we write (x), by that we mean a life aged x.

We can rewrite (1) and (3) as

$$\ell_x = \ell_0\, S(x) = \ell_0\, P(T \geq x) \tag{4}$$

and

$$P(T \geq x) = S(x) = \frac{\ell_x}{\ell_0}. \tag{5}$$

Since people do not live forever, it is reasonable to assume that there is an upper limit for the age. We will denote it by ω. Hence $f(t) = S(t) = 0$ for $t \geq \omega$, therefore, $\ell_x = 0$ for $x \geq \omega$. We will also use the notation

$$d_x = \ell_x - \ell_{x+1}, \tag{6}$$

that is

$$d_x = \ell_0(S(x) - S(x+1)) = \ell_0\, P(x \le T < x+1). \tag{7}$$

Since ℓ_x can be thought of as the expected number of persons surviving to age x and ℓ_{x+1} as the expected number of persons surviving to age $x+1$, d_x can be considered as the expected number of persons dying between the ages of x and $x+1$.

The hazard function, defined by (13) of Section 2.1 is called the force of mortality in actuarial mathematics and is denoted by μ_x:

$$\mu_x = h(x) = -\frac{1}{S(x)}\frac{d}{dx}\, S(x),$$

which can also be written as

$$\mu_x = -\frac{1}{\ell_x}\frac{d}{dx}\,\ell_x = -\frac{d}{dx}\log \ell_x. \tag{8}$$

Of course, (8) does not make sense if $x \ge \omega$. However, we will define $\mu_x = 0$ for $x \ge \omega$ since it makes the formulas more simple. It follows from (14) of Section 2.1 that for any nonnegative n:

$$\ell_{x+n} = \ell_x\, e^{-\int_x^{x+n}\mu_t dt} = \ell_x\, e^{-\int_0^n \mu_{x+t} dt}, \tag{9}$$

and (15) of Section 2.1 implies that

$$\ell_x = \ell_0\, e^{-\int_0^x \mu_t dt}. \tag{10}$$

Using (16) of Section 2.1 we can write the probability density function of T as

$$f(t) = \frac{1}{\ell_0}\,\ell_t\,\mu_t\,. \tag{11}$$

So from (5) and (11) we get

$$\frac{\ell_x}{\ell_0} = P(T \geq x) = \int_x^{\infty} \frac{1}{\ell_0} \ \ell_t \mu_t dt$$

$$= \frac{1}{\ell_0} \int_0^{\infty} \ell_{x+t} \, \mu_{x+t} \, dt. \tag{12}$$

Therefore,

$$\ell_x = \int_x^{\infty} \ell_t \, \mu_t \, dt = \int_0^{\infty} \ell_{x+t} \, \mu_{x+t} \, dt. \tag{13}$$

Furthermore, from (7) and (11) we get

$$\frac{d_x}{\ell_0} = P(x \leq T < x+1) = \int_x^{x+1} \ell_t \, \mu_t \, dt$$

$$= \frac{1}{\ell_0} \int_0^{1} \ell_{x+t} \, \mu_{x+t} \, dt. \tag{14}$$

Thus,

$$d_x = \int_0^{1} \ell_{x+t} \, \mu_{x+t} \, dt. \tag{15}$$

Next, we introduce some more standard actuarial notations.

The probability that a person aged x will survive to the age of $x + n$ is denoted by ${}_np_x$:

$${}_np_x = P(T \geq x + n \mid T \geq x). \tag{16}$$

If $n = 1$, we can drop n from ${}_np_x$. Thus

$$p_x = P(T \geq x + 1 \mid T \geq x). \tag{17}$$

The probability that a person aged x will die before the age of $x + n$ is denoted by ${}_nq_x$:

$${}_nq_x = P(T < x + n \mid T \geq x). \tag{18}$$

Again, if $n = 1$, we can drop n from ${}_nq_x$. So

$$q_x = P(T < x + 1 \mid T \geq x). \tag{19}$$

The probability q_x is also called the rate of mortality at age x.

The probability that a person aged x will survive to the age of $x + m$ but die before the age of $x + m + n$ is denoted by ${}_{m\,|\,n}q_x$. Thus

$$ {}_{m\,|\,n}q_x = P(x + m \le T < x + m + n \mid T \ge x). \tag{20} $$

If $n = 1$, we can drop n from ${}_{m\,|\,n}q_x$. Therefore,

$$ {}_{m\,|}q_x = P(x + m \le T < x + m + 1 \mid T \ge x). \tag{21} $$

It follows from (5) that

$$ {}_np_x = \frac{P(T \ge x + n)}{P(T \ge x)} = \frac{\ell_{x+n}}{\ell_x}, \tag{22} $$

whose special case is

$$ p_x = \frac{\ell_{x+1}}{\ell_x}. $$

We can write

$$ {}_{m+n}p_x = {}_mp_x \;\; {}_np_{x+m} \tag{23} $$

since

$$ \frac{\ell_{x+m+n}}{\ell_x} = \frac{\ell_{x+m}}{\ell_x} \; \frac{\ell_{x+m+n}}{\ell_{x+m}}. $$

We also have

$$ \begin{aligned} {}_nq_x &= \frac{P(x \le T < x + n)}{P(T \ge x)} \\ &= \frac{P(T \ge x) - P(T \ge x + n)}{P(T \ge x)} \\ &= \frac{\ell_x - \ell_{x+n}}{\ell_x}, \end{aligned} \tag{24} $$

and

$$ q_x = \frac{\ell_x - \ell_{x+1}}{\ell_x} = \frac{d_x}{\ell_x}. $$

Finally,

$$
\begin{aligned}
{}_{m|n}q_x &= \frac{P(x + m \le T < x + m + n)}{P(T \ge x)} \\
&= \frac{P(T \ge x + m) - P(T \ge x + m + n)}{P(T \ge x)} \\
&= \frac{\ell_{x+m} - \ell_{x+m+n}}{\ell_x}, \qquad (25)
\end{aligned}
$$

and

$$
{}_{m|}q_x = \frac{\ell_{x+m} - \ell_{x+m+1}}{\ell_x} = \frac{d_{x+m}}{\ell_x}.
$$

We also get

$$
{}_{m|n}q_x = {}_m p_x \times {}_n q_{x+m} \qquad (26)
$$

since

$$
\frac{\ell_{x+m} - \ell_{x+m+n}}{\ell_x} = \frac{\ell_{x+m}}{\ell_x} \times \frac{\ell_{x+m} - \ell_{x+m+n}}{\ell_{x+m}}.
$$

Moreover, we have

$$
{}_{m|n}q_x = {}_m p_x - {}_{m+n}p_x \qquad (27)
$$

since

$$
\frac{\ell_{x+m} - \ell_{x+m+n}}{\ell_x} = \frac{\ell_{x+m}}{\ell_x} - \frac{\ell_{x+m+n}}{\ell_x}.
$$

We can see from (22) and (24) that

$$
{}_n p_x + {}_n q_x = 1. \qquad (28)
$$

The probabilities ${}_n p_x$ and ${}_n q_x$ can be expressed in terms of μ_x, the force of mortality as well. Dividing (9) by ℓ_x, we get

$$
{}_n p_x = e^{-\int_0^n \mu_{x+t} dt}, \qquad (29)
$$

and using (28) we obtain

$$ {}_nq_x = 1 - e^{-\int_0^n \mu_{x+t}dt}. \tag{30}$$

We can obtain different expressions for the probabilities using (11). From (13) and (22) we get

$$ {}_np_x = \frac{1}{\ell_x} \int_{x+n}^{\infty} \ell_t\, \mu_t\, dt = \int_n^{\infty} \frac{\ell_{x+t}}{\ell_x}\, \mu_{x+t}\, dt = \int_n^{\infty} {}_tp_x\, \mu_{x+t}\, dt. \tag{31}$$

From (13) and (24), we obtain

$$ {}_nq_x = \frac{1}{\ell_x} \int_x^{x+n} \ell_t \mu_t dt = \int_0^n \frac{\ell_{x+t}}{\ell_x}\, \mu_{x+t}dt = \int_0^n {}_tp_x\, \mu_{x+t}\, dt, \tag{32}$$

and from (13) and (25) we get

$$ {}_{m|n}q_x = \frac{1}{\ell_x} \int_{x+m}^{x+m+n} \ell_t \mu_t dt = \int_m^{m+n} \frac{\ell_{x+t}}{\ell_x} \mu_{x+t}dt = \int_m^{m+n} {}_tp_x\, \mu_{x+t}dt. \tag{33}$$

We can rewrite (20) of Section 2.1 in terms of q and μ. What we get is that if ε is close to zero then

$$ {}_\varepsilon q_x \approx \varepsilon\, \mu_x, \tag{34}$$

which can be rewritten as

$$ \mu_x \approx \frac{{}_\varepsilon q_x}{\varepsilon}. \tag{35}$$

The expression (34) shows why μ_x is called the force of mortality. If an individual has survived to age x, then the probability of dying in the following ε-long interval is μ_x times the length of the interval.

The probability ${}_tp_x$ can be considered as a function in two variables: $f_0(x,t) = {}_tp_x$. We can ask what the partial derivative of this function is. Because of (22), we can write

$$ \log {}_tp_x = \log \ell_{x+t} - \log \ell_x. \tag{36}$$

Hence,

$$\frac{d}{dt} \log {}_tp_x = \frac{d}{dt} \log \ell_{x+t} . \tag{37}$$

Now,

$$\frac{d}{dt} \log {}_tp_x = \frac{1}{{}_tp_x} \frac{d}{dt} {}_tp_x. \tag{38}$$

Now, using the chain rule and (8) we obtain

$$\frac{d}{dt} \log \ell_{x+t} = \frac{d}{dt} (x+t) \frac{d}{d(x+t)} \log \ell_{x+t} = -\mu_{x+t}. \tag{39}$$

So from (37), (38), and (39) we get

$$\frac{d}{dt} {}_tp_x = -{}_tp_x \, \mu_{x+t}. \tag{40}$$

Moreover, differentiating (36) with respect to x, we obtain

$$\frac{d}{dx} \log {}_tp_x = \frac{d}{dx} \log \ell_{x+t} - \frac{d}{dx} \log \ell_x. \tag{41}$$

Expanding the terms in (41), we get

$$\frac{d}{dx} \log {}_tp_x = \frac{1}{{}_tp_x} \frac{d}{dx} {}_tp_x, \tag{42}$$

$$\frac{d}{dx} \log \ell_{x+t} = -\mu_{x+t}, \tag{43}$$

and

$$\frac{d}{dx} \log \ell_x = -\mu_x. \tag{44}$$

Thus,

$$\frac{d}{dx} {}_tp_x = {}_tp_x(\mu_x - \mu_{x+t}). \tag{45}$$

As we have already mentioned in Section 2.1, when an insurance policy is issued to an individual, we are interested in the future lifetime of the insured. Therefore, it is useful to introduce a random variable T_x for all

ages x, defined as the future lifetime beyond the age of x given the individual has survived to age x:

$$T_x = T - x \mid T \geq x.$$

In other words, $T_x \geq 0$ and for any $t \geq 0$

$$P(T_x \leq t) = \frac{P(x \leq T \leq x + t)}{P(T \geq x)} = P(T \leq x + t \mid T \geq x).$$

Therefore, the distribution function of T_x at t is

$$\begin{aligned} F_x(t) &= P(T_x \leq t) \\ &= P(T \leq x + t \mid T \geq x) \\ &= {}_tq_x = 1 - {}_tp_x \end{aligned} \tag{46}$$

and using (16), we can express the survival function of T_x at t as

$$\begin{aligned} S_x(t) &= P(T_x > t) \\ &= P(T - x > t \mid T \geq x) \\ &= {}_tp_x \,. \end{aligned} \tag{47}$$

From (20) we get

$$\begin{aligned} P(m \leq T_x < m + t) &= P(m \leq T - x < m + t \mid T \geq x) \\ &= {}_{m|t}q_x. \end{aligned} \tag{48}$$

We can obtain the probability density function of T_x by differentiating $F_x(t)$ with respect to t. If we use (40) and (46), we get

$$\begin{aligned} \frac{d}{dt} F_x(t) &= \frac{d}{dt} (1 - {}_tp_x) \\ &= {}_tp_x \, \mu_{x+t}. \end{aligned}$$

Therefore, the probability density function of T_x is

$$f_x(t) = {}_tp_x \, \mu_{x+t} \,. \tag{49}$$

We may be interested in the expected future lifetime of an individual aged x. Denoting it by $\mathring{e}_x$, we have $\mathring{e}_x = E(T_x)$ and using (49) we get

$$\mathring{e}_x = \int_0^\infty t \, {}_tp_x \, \mu_{x+t} \, dt.$$

In order to simplify this integral, we can use integration by parts. Let $u(t) = t$ and $v'(t) = {}_tp_x \times \mu_{x+t}$. Then $u'(t) = 1$, and from (40), we get $v(t) = -{}_tp_x$. So the integral can be rewritten as

$$-{}_tp_x \cdot t \,\Big|_{t=0}^{\infty} + \int_0^{\infty} {}_tp_x dt = \int_0^{\infty} {}_tp_x \, dt,$$

thus

$$\overset{\circ}{e}_x = E(T_x) = \int_0^{\infty} {}_tp_x \, dt. \tag{50}$$

We may also ask what the variance of the future lifetime random variable is. To find this, we need to determine $E(T_x^2)$ first. Using (49), we get

$$E(T_x^2) = \int_0^{\infty} t^2 \, {}_tp_x \, \mu_{x+t} \, dt.$$

We can use integration by parts again with the choice of $u(t) = t^2$ and $v'(t) = {}_tp_x \, \mu_{x+t}$. Then $u'(t) = 2t$, and $v(t) = -{}_tp_x$. Thus,

$$\begin{aligned} E(T_x^2) &= -{}_tp_x \cdot t^2 \,\Big|_{t=0}^{\infty} + 2\int_0^{\infty} t \, {}_tp_x \, dt \\ &= 2\int_0^{\infty} t \, {}_tp_x \, dt. \end{aligned}$$

Since $V(T_x) = E(T_x^2) - (E(T_x))^2$, we get

$$V(T_x) = 2\int_0^{\infty} t \, {}_tp_x \, dt - \overset{\circ}{e}{}_x^2. \tag{51}$$

In many problems of life insurance, the exact age at death is not important. It is possible that we are only interested in the number of complete one year long periods an individual survives. For example, if the insured gets a sum of \$2000 each time he/she reaches a birthday between the ages of 60 and 70, it does not make a difference whether a death occurs at the exact age of 62.1 or 62.8. The only thing that counts is that the death occurs between the ages of 62 and 63. Hence, it is worth studying the

number of complete one year long periods an individual aged x survives, which is called the curtate future lifetime. So we introduce the random variable K_x, and define it as the integer part of the future lifetime beyond the age of x given the individual has survived to age x. That is,

$$K_x = [T - x] \mid T \geq x = [T_x] \tag{52}$$

where [ ] is the integer part function, that means $[z]$ is the largest integer less than or equal to z.

Now K_x is a discrete random variable, and its probability function can be obtained using (48):

$$\begin{aligned} P(K_x = k) &= P([T_x] = k) \\ &= P(k \leq T_x < k + 1) \\ &= {}_{k|}q_x, \text{ for } k = 0,1,2,\dots . \end{aligned} \tag{53}$$

Using (46), the distribution function of K_x can be obtained as

$$\begin{aligned} P(K_x \leq k) &= P([T_x] \leq k) \\ &= P(T_x < k + 1) \\ &= P[T_x \leq k + 1] \\ &= {}_{k+1}q_x, \end{aligned} \tag{54}$$

and therefore, the survival function of K_x at k equals

$$P(K_x > k) = {}_{k+1}p_x.$$

Note that since K is a discrete random variable, we must pay attention whether an inequality involving K_x is strict or not.

Now, let us find the expected curtate future lifetime and its variance.

The expected curtate future lifetime of an individual aged x is denoted by e_x. Then $e_x = E(K_x)$ and using (27) and (53) we obtain

$$\begin{aligned} e_x &= \sum_{k=0}^{\infty} k \, P(K_x = k) \\ &= \sum_{k=0}^{\infty} k \, {}_{k|}q_x \\ &= \sum_{k=0}^{\infty} k({}_kp_x - {}_{k+1}p_x) \\ &= \sum_{k=0}^{\infty} k \, {}_kp_x - \sum_{m=1}^{\infty} (m - 1) \, {}_mp_x \end{aligned}$$

$$= \sum_{k=1}^{\infty} k \, {}_kp_x - \sum_{k=1}^{\infty} (k-1) \, {}_kp_x$$

$$= \sum_{k=1}^{\infty} {}_kp_x. \tag{55}$$

In order to find the variance of the curtate future lifetime random variable, we need to determine $E(K_x^2)$ first. Following the steps of (55) we obtain

$$E(K_x^2) = \sum_{k=0}^{\infty} k^2 \, P(K_x = k)$$

$$= \sum_{k=1}^{\infty} k^2 \, {}_kp_x - \sum_{k=1}^{\infty} (k-1)^2 \, {}_kp_x.$$

Now, $k^2 - (k-1)^2 = 2k - 1$, thus

$$E(K_x^2) = 2 \sum_{k=1}^{\infty} k \, {}_kp_x - \sum_{k=1}^{\infty} {}_kp_x$$

$$= 2 \sum_{k=1}^{\infty} k \, {}_kp_x - e_x, \tag{56}$$

where we used (55). Therefore,

$$V(K_x) = 2 \sum_{k=1}^{\infty} k \, {}_kp_x - e_x - e_x^2 . \tag{57}$$

The question arises here how we can obtain the numerical values of the mortality functions. One possibility is to use a functional form of ℓ_x. Usually, a function for the rate of mortality μ_x is given from which ℓ_x can be obtained using (10). The following expressions define some of the most popular analytical laws of mortality:

De Moivre's law: $\mu_x = \dfrac{1}{\omega - x}$,

Gompertz's law: $\mu_x = B \, C^x$,

Makeham's law: $\mu_x = A + B \, C^x$,

double geometric law: $\mu_x = A + B\,C^x + Mn^x$,

Makeham's second law: $\mu_x = A + H\,x + B \cdot C^x$,

Perk's law: $\mu_x = \dfrac{A + B\,C^x}{K\,C^{-x} + 1 + D \cdot C^x}$,

where $A, B, C, D, H, K, M,$ and n are constants.

Unfortunately, all the analytical laws of mortality proposed so far are either two complicated to use or they cannot describe the real mortality experience with sufficient accuracy over a wider range of ages. However, they were extensively used before the invention of computers.

The second possibility of obtaining mortality data is to tabulate values in a table. These are called mortality tables. We will study them in the next section.

EXAMPLE 2.1. The force of mortality for a person is 0.002 between the ages of 30 and 40. Find the probability that a person aged 30 will

a) survive the following year.
b) die within one year.
c) survive to age 35.
d) die before the age of 38.
e) die between the ages of 35 and 40.

Solution: We have

$$\mu_x = 0.002 \text{ if } 30 \le x \le 40.$$

From (29), we get

$$_np_x = e^{-\int_0^n 0.002dt} = e^{-0.002n}, \text{ for } 30 \le x \le x + n \le 40.$$

Thus we obtain the following results:

a) The probability that a person aged 30 will survive to age 31 is

$$p_{30} = e^{-0.002 \cdot 1} = 0.9980020.$$

b) The probability that a person aged 30 will die before the age of 31 is

$$q_{30} = 1 - p_{30} = 1 - 0.9980020 = 0.0019980.$$

c) The probability that a person aged 30 will survive to age 35 is

$$_5p_{30} = e^{-0.002 \cdot 5} = 0.9900498.$$

d) The probability that a person aged 30 will die before the age of 38 is

$$_8q_{30} = 1 - {_8p_{30}} = 1 - e^{-0.002 \cdot 8} = 0.0158727.$$

e) The probability that a person aged 30 will die between the ages of 36 and 40 is

$$_{6|4}q_{30} = {_6p_{30}} - {_{10}p_{30}} = e^{-0.002 \cdot 6} - e^{-0.002 \cdot 10} = 0.0078730.$$

PROBLEMS

2.1. The force of mortality for a person is 0.003 between the ages of 35 and 45. Find the probability that a person aged 35 will

a) survive the following year.
b) die before reaching the age of 36.
c) survive to age 42.
d) die within 10 years.
e) die between the ages of 39 and 44.

2.2. The force of mortality for a person is 0.004 between the ages of 40 and 50 and 0.01 between the ages of 50 and 60. Determine the probability that a person aged 40 will

a) die before the age of 45.
b) survive to age 50.
c) survive to age 60.
d) die between the ages of 45 and 55.
e) die between the ages of 52 and 55.

2.3. Prove that if the force of mortality follows De Moivre's law then

$$\ell_x = \frac{\ell_0}{\omega} (\omega - x).$$

That means, the original population of ℓ_0 at age zero decreases linearly to 0 at age ω.

2.3. MORTALITY TABLES

Mortality tables (also called life tables) describe the mortality experience of a certain group of people. Each country has its own mortality tables. If the mortality experience of the total population of a country is taken into account, we are speaking about a population mortality table. There are also separate mortality tables for men and women. Some mortality tables are based solely on the past mortality experience of insured lives.

Mortality tables usually give the values of ℓ_x only for integer values of x. See the table in Appendix 2. We will use this table throughout the book.

The lack of knowledge of ℓ_x for non-integer x does not cause any great difficulty, since in the practice of life insurance, it is usually sufficient to use the integer part of the age of the insured. For example, age can be taken as the difference between the year of the purchase of the policy and the year of birth. Proceeding this way, we will underestimate the exact age for some policyholders and overestimate it for others, but if an insurance company has a large number of policies, these effects will balance each other. If it still becomes necessary to use the exact age, we can use interpolation to approximate ℓ_x for non-integer x. We will discuss this question later in this section.

Appendix 2 does not only contain the values of ℓ_x, it also tabulates d_x and q_x. Since the numerical value of q_x is often very small, between 0.001 and 0.01, the table gives the values of $1000q_x$ from which q_x can be obtained by shifting the decimal point three digits to the left.

Let us compute some probabilities using this table.

The probability that a person aged 40 survives the following year is

$$p_{40} = \frac{\ell_{41}}{\ell_{40}} = \frac{92872.62}{93131.64} = 0.9972188$$

which can also be obtained as

$$p_{40} = 1 - q_{40} = 1 - 0.0027812 = 0.9972188.$$

The probability that a person aged 50 survives to age 70 is

$$_{20}p_{50} = \frac{\ell_{70}}{\ell_{50}} = \frac{66161.54}{89509.00} = 0.7391608.$$

The probability that a person aged 80 will die within one year is

$$q_{80} = \frac{\ell_{80} - \ell_{81}}{\ell_{80}} = \frac{39143.64 - 36000.37}{39143.64} = \frac{3143.27}{39143.64} = 0.0803009.$$

This can also be computed as

$$q_{80} = \frac{d_{80}}{\ell_{80}} = \frac{3143.2679}{39143.64} = 0.0803009.$$

The result can also be obtained directly from the q_x column of the table.

The probability that a person aged 55 will die within 5 years is

$$\begin{aligned} {}_5q_{55} &= \frac{\ell_{55} - \ell_{60}}{\ell_{55}} \\ &= \frac{86408.60 - 81880.73}{86408.60} \\ &= 0.0524007. \end{aligned}$$

The probability that a person aged 40 will die between the ages of 70 and 80 is

$$\begin{aligned} {}_{30|10}q_{40} &= \frac{\ell_{70} - \ell_{80}}{\ell_{40}} \\ &= \frac{66161.54 - 39143.64}{93131.64} \\ &= 0.2901044. \end{aligned}$$

The probability that a person aged 70 will die between the ages of 74 and 75 is

$$\begin{aligned} {}_{4|}q_{70} &= \frac{\ell_{74} - \ell_{75}}{\ell_{70}} \\ &= \frac{56640.51 - 53960.80}{66161.54} \\ &= 0.0405025. \end{aligned}$$

The last probability can also be obtained by writing

$$\begin{aligned} {}_{4|}q_{70} &= \frac{d_{74}}{\ell_{70}} \\ &= \frac{2679.7050}{66161.54} \\ &= 0.0405025. \end{aligned}$$

Next, we want to focus on what we can say about the value of ℓ_x for non-integer x. Of course, we cannot determine the exact value of ℓ_x for fractional x, so we have to turn to some approximation method. A simple but useful method is the linear approximation. If x is an integer then we can look up the values of ℓ_x and ℓ_{x+1} in the table. Applying a linear interpolation for the interval between x and $x + 1$, we get

$$\ell_{x+t} \approx \ell_x + t(\ell_{x+1} - \ell_x) = \ell_x - t\, d_x \tag{1}$$

or equivalently

$$\ell_{x+t} \approx (1 - t)\, \ell_x + t\, \ell_{x+1} \text{ for } x \text{ integer and } 0 \le t \le 1. \tag{2}$$

For example, using Appendix 2, $\ell_{20.4}$ can be approximated as

$$\ell_{20.4} \approx (1 - 0.4)\, \ell_{20} + 0.4\, \ell_{21} = 0.6(96178.01) + 0.4(96078.95) = 96138.39.$$

Let us recall that at the beginning of Section 2.2 we interpreted ℓ_x as the number of people who survive to age x from an original population of ℓ_0 at the moment of birth. In this context, the linear interpolation of ℓ_x means that the distribution of the d_x deaths between the integer ages x and $x + 1$ is assumed to be uniform.

Using (2), we can approximate the probabilities as well. We get

$${}_tp_x = \frac{\ell_{x+t}}{\ell_x} \approx \frac{(1 - t)\, \ell_x + t\, \ell_{x+1}}{\ell_x} = 1 - t + tp_x = 1 - t(1 - p_x) = 1 - tq_x \tag{3}$$

and

$${}_tq_x \approx 1 - {}_tp_x = 1 - (1 - tq_x) = tq_x \text{ for } x \text{ integer and } 0 \le t \le 1. \tag{4}$$

For example, using Appendix 2, we get

$${}_{0.7}p_{80} \approx 1 - 0.7\, q_{80} = 1 - 0.7(0.0803009) = 0.9437894$$

and

$${}_{0.9}q_{60} \approx 0.9\, q_{60} = 0.9(0.0137604) = 0.0123844.$$

If t is fractional, but greater than 1, we have to split the probabilities into two parts, before we can use the linear interpolation, as the following computations show:

$$_{10.5}p_{40} = {}_{10}p_{40} \cdot {}_{0.5}p_{50} \approx \frac{\ell_{50}}{\ell_{40}}(1 - 0.5\, q_{50}) = \frac{89509.00}{93131.64}[1 - 0.5(0.0059199)]$$
$$= 0.9582571$$

and

$$_{4.2}q_{50} = 1 - {}_{4.2}p_{50} = 1 - {}_{4}p_{50}\, {}_{0.2}p_{54} \approx 1 - \frac{\ell_{54}}{\ell_{50}}(1 - 0.2\, q_{54})$$
$$= 1 - \frac{87126.20}{89509.00}[1 - 0.2(0.0082364)] = 0.0282242.$$

If we want to approximate the force of mortality, (2) is not appropriate to use. This is easy to see since the definition of μ_x in (8) of Section 2.2 assumes that ℓ_x is differentiable, but the ℓ_x function obtained by linear interpolation is not differentiable at integer x. The interested reader may consult A. Neil: *Life Contingencies, 1989* for approximation methods applicable here.

Next, we show how the expected future lifetime and the expected curtate future lifetime can be obtained from a mortality table.

Using (55) of Section 2.2, the expected curtate future lifetime is

$$e_x = \sum_{k=1}^{\infty} {}_kp_x = \sum_{k=1}^{\infty} \frac{\ell_{x+k}}{\ell_x} = \frac{\sum_{k=1}^{\infty} \ell_{x+k}}{\ell_x} = \frac{\ell_{x+1} + \ell_{x+2} + \ldots}{\ell_x} \quad \text{for } x \text{ integer.} \tag{5}$$

For example, using the mortality table in Appendix 2, we get

$$e_0 = \frac{\sum_{k=1}^{110} \ell_k}{\ell_0} = 71.31$$

and

$$e_{20} = \frac{\sum_{k=1}^{110} \ell_{20+k}}{\ell_{20}} = 53.96.$$

We can compute the exact value of e_x from a mortality table since $\ell_x, \ell_{x+1}, \ell_{x+2}, \ldots$ are known values. On the other hand, if we want to find the expected future lifetime, we have

$$\mathring{e}_x = \int_0^\infty {}_tp_x \, dt$$

from (50) of Section 2.2, so we need the value of ${}_tp_x$ for non-integer t's as well. We can write

$$\mathring{e}_x = \sum_{k=0}^\infty \int_k^{k+1} {}_tp_x \, dt = \sum_{k=0}^\infty {}_kp_x \int_0^1 {}_tp_{x+k} \, dt,$$

and using approximation (3) we have

$$\int_0^1 {}_tp_{x+k} \, dt \approx \int_0^1 (1 - t\, q_{x+k}) dt = 1 - \frac{q_{x+k}}{2}.$$

Therefore,

$$\mathring{e}_x \approx \sum_{k=0}^\infty {}_kp_x \left(1 - \frac{q_{x+k}}{2}\right) = \sum_{k=0}^\infty {}_kp_x - \frac{1}{2} \sum_{k=0}^\infty {}_{k|}q_x.$$

Now,

$$\sum_{k=0}^\infty {}_kp_x = {}_0p_x + \sum_{k=1}^\infty {}_kp_x = 1 + e_x$$

and

$$\sum_{k=0}^\infty {}_{k|}q_x = \sum_{k=0}^\infty ({}_kp_x - {}_{k+1}p_x) = \sum_{k=0}^\infty {}_kp_x - \sum_{\ell=1}^\infty {}_\ell p_x = {}_0p_x = 1.$$

Thus, the approximation for the expected future lifetime is

$$\mathring{e}_x \approx 1 + e_x - \frac{1}{2} = e_x + \frac{1}{2} = \frac{\sum_{k=1}^\infty \ell_{x+k}}{\ell_x} + \frac{1}{2} = \frac{\ell_{x+1} + \ell_{x+2} + \dots}{\ell_x} + \frac{1}{2}. \quad (6)$$

For example, using the mortality table in Appendix 2, we get

$$\mathring{e}_0 \approx \frac{\sum_{k=1}^{110} \ell_k}{\ell_0} + \frac{1}{2} = 71.81,$$

$$\mathring{e}_1 \approx \frac{\sum_{k=1}^{109} \ell_{1+k}}{\ell_1} + \frac{1}{2} = 72.29,$$

$$\mathring{e}_2 \approx 71.39,$$

$$\mathring{e}_3 \approx 70.48,$$
$$\mathring{e}_{20} \approx 54.46,$$

$$\mathring{e}_{50} \approx 27.09.$$

It may seem surprising that the expected future lifetime is less at the time of birth than at the age of 1. The reason for that is the extremely high probability of dying in the first year: $q_0 = 0.024217$. Once a baby survives his/her first year, the life expectations improve dramatically. Indeed, q_x drops sharply after the first year: $q_1 = 0.0013431$, $q_2 = 0.0012237$, etc. and the value of q_0 is not reached again before the age of 65.

If we consider a new entrant to a life insurance, his/her probability of dying in the first years of the policy tends to be lower than what can be expected for other people of the same age. This has two main reasons. On the one hand, if the benefit of a life insurance is payable on death, the major risk for the insurance company is that the insured dies too soon, since then the premium will not accumulate to a high enough level. Therefore, in the underwriting process, it will be checked whether the applicant is in a reasonably good health condition. If certain standards set by the company are not met, the application for the insurance will be turned down. Thus, at the time the policy is issued, the mortality of the policyholder is lower than that of other people of the same age. On the other hand, people do not buy life annuities unless they are in good health, since otherwise they cannot expect to receive the annuity payments for long. So this self selection also results in mortality values lower than average. However, after a couple of years, this initial selection effect wears off.

Mortality tables that reflect the selection effect are called select mortality tables. The length of the period from entry for which a select mortality table defines special survival probabilities is called the select period. Once the end of the select period is reached, the mortality is not influenced by the duration of the policy any more. So if a policy is already

in force longer than the select period, the mortality can be calculated from a regular mortality table, called the ultimate mortality table.

There are select tables with different select periods. The length of the select period is usually 3, 5, 10, or 15 years. We will denote the length of the select period by r (measured in years).

If we use a select mortality table, the symbol $[x]$ is used to denote a life aged x at the commencement of the insurance.

The probability a life $[x]$ survives to age $x + k + 1$, under the condition he/she survives to age $x + k$ is denoted by $p_{[x]+k}$. However, if $[x]$ has survived to age $x + r$, the future mortality does not depend on the duration of the insurance any more. Therefore, if $x_1 + k_1 = x_2 + k_2$ and $k_1, k_2 \geq r$, we get

$$p_{[x_1]+k_1} = p_{[x_2]+k_2}.$$

Thus, if $k \geq r$ then the probability of survival depends on x and k only through $x + k$. So we can use the notation p_{x+k} instead of $p_{[x]+k}$ if $k \geq r$. The probability $p_{[x]+k}$ $(k < r)$ is called a select value while p_{x+k} $(k \geq r)$ is an ultimate value.

The probability a life $[x]$ dies before age $x + k + 1$, given he/she survives to age $x + k$ is denoted by $q_{[x]+k}$, which can be simplified as q_{x+k} if $k \geq r$.

The usual relationship between the probability of death and the probability of survival holds here, too:

$$p_{[x]+k} + q_{[x]+k} = 1. \tag{7}$$

A part of a select mortality table with a select period of 3 years is given in Table 1 (it is not related to the table in Appendix 2). From this table, we get the select values

$$q_{[25]} = 0.00140,$$

$$q_{[25]+1} = 0.00171,$$

and

$$q_{[25]+2} = 0.00196.$$

Furthermore,

$$q_{[25]+3} = q_{25+3} = q_{28} = 0.00212,$$

$$q_{[25]+4} = q_{25+4} = q_{29} = 0.00217,$$

and

TABLE 1
SECTION OF SELECT AND ULTIMATE TABLE

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	$x+3$
20	946,394	945,145	943,671	942,001	23
21	944,710	943,435	941,916	940,202	24
22	942,944	941,652	940,108	938,359	25
23	941,143	939,835	938,265	936,482	26
24	939,279	937,964	936,379	934,572	27
25	937,373	936,061	934,460	932,628	28
26	935,433	934,123	932,507	930,651	29
27	933,467	932,151	930,520	928,631	30
28	931,488	930,156	928,491	926,560	31
29	929,476	928,119	926,421	924,429	32
30	927,422	926,040	924,290	922,220	33
$[x]$	$d_{[x]}$	$d_{[x]+1}$	$d_{[x]+2}$	d_{x+3}	$x+3$
20	1,249	1,474	1,670	1,799	23
21	1,275	1,519	1,714	1,843	24
22	1,292	1,544	1,749	1,877	25
23	1,308	1,570	1,783	1,910	26
24	1,315	1,585	1,807	1,944	27
25	1,312	1,601	1,832	1,977	28
26	1,310	1,616	1,856	2,020	29
27	1,316	1,631	1,889	2,071	30
28	1,332	1,665	1,931	2,131	31
29	1,357	1,698	1,992	2,209	32
30	1,382	1,750	2,070	2,306	33
$[x]$	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	$x+3$
20	.00132	.00156	.00177	.00191	23
21	.00135	.00161	.00182	.00196	24
22	.00137	.00164	.00186	.00200	25
23	.00139	.00167	.00190	.00204	26
24	.00140	.00169	.00193	.00208	27
25	.00140	.00171	.00196	.00212	28
26	.00140	.00173	.00199	.00217	29
27	.00141	.00175	.00203	.00223	30
28	.00143	.00179	.00208	.00230	31
29	.00146	.00183	.00215	.00239	32
30	.00149	.00189	.00224	.00250	33

Source: From Table 3 of *Life Contingencies* by C. W. Jordan, Jr. (2nd Edition), page 26.

$$q_{[25]+5} = q_{25+5} = q_{30} = 0.00223$$

etc. are ultimate values. Note that

$$q_{[25]} = 0.00140 < q_{25} = 0.00200$$

$$q_{[25]+1} = 0.00171 < q_{26} = 0.00204$$

and

$$q_{[25]+2} = 0.00196 < q_{27} = 0.00208$$

showing that people who have recently purchased an insurance have lower mortality rates indeed.

In a select mortality table, values for $\ell_{[x]+k}$ can also be found. They are defined in a way such that they satisfy the equation

$$p_{[x]+k} = \frac{\ell_{[x]+k+1}}{\ell_{[x]+k}}. \tag{8}$$

If $k \geq r$, then (8) becomes

$$p_{x+k} = \frac{\ell_{x+k+1}}{\ell_{x+k}} = \frac{\ell_{y+1}}{\ell_y} \tag{9}$$

with $y = x + k$. The values of ℓ_y are given in the column headed ℓ_{x+r}. For example, in Table 1, we find

$$\ell_{25} = 938359,$$

and

$$\ell_{30} = 928631.$$

The ℓ's standing in the column headed ℓ_{x+r} are called the ultimate values of ℓ and they have the same properties as in any regular mortality table.

If the column headed ℓ_{x+r} is given, the rest of the table can be obtained using (8). Note that (8) can be rewritten as

$$\ell_{[x]+k} = \frac{\ell_{[x]+k+1}}{p_{[x]+k}}$$

or

$$\ell_{[x]+k} = \frac{\ell_{[x]+k+1}}{1 - q_{[x]+k}}, \quad k = 0,1,\ldots,r-1.$$

So if the values of $q_{[x]+k}$ are known, we can obtain $\ell_{[x]+k}$ using a recursion:

$$\ell_{[x]+r-1} = \frac{\ell_{x+r}}{1 - q_{[x]+r-1}},$$

$$\ell_{[x]+r-2} = \frac{\ell_{[x]+r-1}}{1 - q_{[x]+r-2}},$$

$$\vdots$$

$$\ell_{[x]+1} = \frac{\ell_{[x]+2}}{1 - q_{[x]+1}},$$

and

$$\ell_{[x]} = \frac{\ell_{[x]+1}}{1 - q_{[x]}}.$$

Thus, we can determine $\ell_{[x]+k}$ explicitly as

$$\ell_{[x]+k} = \frac{\ell_{x+r}}{\prod_{j=k}^{r-1} (1 - q_{[x]+j})}. \tag{10}$$

For example, using the mortality rates of Table 1, based on the value

$$\ell_{28} = 932628,$$

we can derive

$$\ell_{[25]+2} = \frac{\ell_{28}}{1 - q_{[25]+2}} = \frac{932628}{1 - 0.00196} = 934460,$$

$$\ell_{[25]+1} = \frac{\ell_{[25]+2}}{1 - q_{[25]+1}} = \frac{934460}{1 - 0.00171} = 936061,$$

and

$$\ell_{[25]} = \frac{\ell_{[25]+1}}{1 - q_{[25]}} = \frac{936061}{1 - 0.00140} = 937373.$$

These are exactly the values standing in the table for

$$\ell_{[25]+k} \ (k = 0,1,2).$$

From (7) and (8) we get

$$q_{[x]+k} = 1 - p_{[x]+k} = 1 - \frac{\ell_{[x]+k+1}}{\ell_{[x]+k}} = \frac{\ell_{[x]+k} - \ell_{[x]+k+1}}{\ell_{[x]+k}}, \quad (11)$$

so introducing

$$d_{[x]+k} = \ell_{[x]+k} - \ell_{[x]+k+1}, \quad (12)$$

we can rewrite (11) as

$$q_{[x]+k} = \frac{d_{[x]+k}}{\ell_{[x]+k}}. \quad (13)$$

Values of $d_{[x]+k}$ $(k = 0,1,2)$ can also be found in Table 1.

We can give the mortality functions $\ell_{[x]+k}$ and $d_{[x]+k}$ the following interpretation. Consider a group of $\ell_{[x]}$ people taking out an insurance at the age of x. Then the expected number of survivors to age $x + k$ is $\ell_{[x]+k}$ and the expected number of deaths between the ages of $x + k$ and $x + k + 1$ is $d_{[x]+k}$.

It is important to remember that if we pick one column of a select table headed $\ell_{[x]+k}$ with $k < r$, the ℓ's standing in this column are not related to each other. For example, $p_{[x]}$ is not equal to $\ell_{[x]}$ divided by $\ell_{[x+1]}$. Dividing ℓ's of the same column only makes sense if they belong to the column of ultimate values as (9) shows.

Using the values of $\ell_{[x]+k}$, we can express probabilities conveniently.

The probability that a life $[x]$ will survive to age $x + n + k$ under the condition he/she survives to age $x + k$ is denoted by ${}_np_{[x]+k}$. Since

$${}_np_{[x]+k} = p_{[x]+k} \cdot p_{[x]+k+1} \cdot \cdots \cdot p_{[x]+k+n-1}$$

$$= \frac{\ell_{[x]+k+1}}{\ell_{[x]+k}} \cdot \frac{\ell_{[x]+k+2}}{\ell_{[x]+k+1}} \cdot \ldots \cdot \frac{\ell_{[x]+k+n}}{\ell_{[x]+k+n-1}},$$

we get

$$ {}_n p_{[x]+k} = \frac{\ell_{[x]+k+n}}{\ell_{[x]+k}}. \tag{14} $$

The probability that a life $[x]$ will die before the age $x + k + n$ given he/she survives to age $x + k$ is denoted by ${}_n q_{[x]+k}$. We have

$$ {}_n q_{[x]+k} = 1 - {}_n p_{[x]+k} = 1 - \frac{\ell_{[x]+k+n}}{\ell_{[x]+k}} = \frac{\ell_{[x]+k} - \ell_{[x]+k+n}}{\ell_{[x]+k}}. \tag{15} $$

The probability that a life $[x]$ will die between the ages $x + k + m$ and $x + k + m + n$ under the condition he/she survives to age $x + k$ is denoted by ${}_{m|n} q_{[x]+k}$. We have

$$ {}_{m|n} q_{[x]+k} = {}_m p_{[x]+k} - {}_{m+n} p_{[x]+k} = \frac{\ell_{[x]+k+m} - \ell_{[x]+k+m+n}}{\ell_{[x]+k}}. \tag{16} $$

In particular, if $n = 1$, we can drop n from the prefix and write

$$ {}_{m|} q_{[x]+k} = \frac{\ell_{[x]+k+m} - \ell_{[x]+k+m+1}}{\ell_{[x]+k}} = \frac{d_{[x]+k+m}}{\ell_{[x]+k}}. \tag{17} $$

For example, using Table 1, we obtain

$$ {}_2 p_{[20]} = \frac{\ell_{[20]+2}}{\ell_{[20]}} = \frac{943671}{946394} = 0.99712, $$

$$ {}_5 q_{[25]+2} = 1 - \frac{\ell_{32}}{\ell_{[25]+2}} = 1 - \frac{924429}{934460} = 0.01073, $$

$$ {}_{2|4} q_{[23]} = \frac{\ell_{[23]+2} - \ell_{29}}{\ell_{[23]}} = \frac{938265 - 930651}{941143} = 0.00809, $$

and

$$ {}_{6|} q_{[20]+4} = \frac{d_{30}}{\ell_{24}} = \frac{2071}{940202} = 0.00220. $$

We can also derive equations similar to (23) of Section 2.2, (26) of Section 2.2, and (27) of Section 2.2. Since

$$\frac{\ell_{[x]+k+m+n}}{\ell_{[x]+k}} = \frac{\ell_{[x]+k+m}}{\ell_{[x]+k}} \cdot \frac{\ell_{[x]+k+m+n}}{\ell_{[x]+k+m}},$$

we get

$$_{m+n}p_{[x]+k} = {}_{m}p_{[x]+k} \cdot {}_{n}p_{[x]+k+m}. \tag{18}$$

Furthermore, we get

$$_{m|n}q_{[x]+k} = {}_{m}p_{[x]+k} \cdot {}_{n}q_{[x]+k+m} \tag{19}$$

and

$$_{m|n}q_{[x]+k} = {}_{m}p_{[x]+k} - {}_{m+n}p_{[x]+k}. \tag{20}$$

The proofs of (19) and (20) are left to the reader.

In the remainder of the book, we will use the mortality table in Appendix 2.

PROBLEMS

3.1. Using the mortality table in Appendix 2, obtain

a) p_{50}
b) q_{30}
c) $_{7}p_{45}$
d) $_{45}q_{20}$
e) $_{10|5}q_{50}$

3.2. Based on the mortality table in Appendix 2, find the probability of the following events.

a) A life aged 25 dies before the age of 30.
b) A life aged 30 survives to age 60.
c) A life aged 40 dies within two years.
d) A life aged 55 dies between the ages of 60 and 80.

3.3. Based on the mortality table in Appendix 2, use linear interpolation to calculate

a) $\ell_{30.7}$

b) $_{0.4}p_{35}$
c) $_{6.9}p_{45}$
d) $_{2.3}q_{70}$

3.4. Using the mortality table in Appendix 2, obtain

a) e_{30}
b) $\mathring{e}_{30}$
c) e_{40}
d) $\mathring{e}_{45}$

3.5. Using Table 1, obtain

a) $q_{[20]}$
b) $p_{[20]+2}$
c) $p_{[20]+3}$
d) $_{5}p_{[20]+1}$
e) $_{6}q_{[20]+2}$
f) $_{3|4}q_{[20]}$

3.6. Prove (19) and (20).

CHAPTER 3

LIFE INSURANCES AND ANNUITIES

In Chapter 1, when we discussed some aspects of financial mathematics, the cash flows we studied consisted of payments whose timing and size were fixed in advance. However, we have already pointed out that it is possible to make payments dependable on death or survival of a person. For example, a retired employee may receive a regular payment each month until his/her death. If we want to find out how much the employer had to pay into a pension fund in order to provide the employer with this life annuity, we cannot use the techniques of Chapter 1. Those methods can only be used if we know for sure how many payments will be made. However, here we have a situation where the exact number of payments cannot be foreseen. The valuation of a life annuity cannot be made without using the theory of mortality.

As another example, consider an insurance that pays a sum on death of the insured. These insurances are called life insurances. It is clear again that we need to use a mortality assumption or a mortality table if we want to determine the price of this insurance.

In general, we can say that the theory of life insurance is a combination of financial mathematics and the theory of mortality. In this chapter, we will always assume that the mortality of the insured life is known, and the insurance company can invest money at a fixed interest rate of i per annum.

3.1. Stochastic Cash Flows

Assume a person aged x takes out an insurance at time t_0. The insurance will pay him/her certain sums at certain times depending on his/her death or survival. Since the premiums are paid by the insured, their payment is also contingent on survival. So both the benefit and the premium payments have to be treated as the components of a stochastic cash flow. Of course, the benefit and the premium payments must be given different signs. If the payment of an amount is contingent on survival, it is called a survival benefit and if a sum is payable on death, it is called a death benefit. Premium payments can be treated as negative survival benefits. Moreover, the expenses incurred by the insurance company can also be included in the cash flow with positive signs. Expenses will be discussed in Chapter 4.

Since the payments of an insurance depend on death or survival, if we know the time the insured dies (say at time $t_0 + t$), we can say exactly what payments are made under that particular policy. For example, in the case of a life annuity, only those payments are made which are due between times t_0 and $t_0 + t$. In the case of a death benefit, the time of death determines the payment of the death benefit. So in general, the cash flow of an

insurance can be described by a function in two variables: $C(t^*,t)$, where $C(t^*,t)$ is the payment at time $t_0 + t^*$ if death occurs at time $t_0 + t$ ($t \geq 0$, $t^* \geq 0$). If the payment is made continuously, we have to use the function $\rho(t^*,t)$ which gives the rate of payment per annum at time $t_0 + t^*$ if death occurs at time $t_0 + t$.

Let us see some examples of the functions $C(t^*,t)$ and $\rho(t^*,t)$.

First, assume the insurance pays an amount of C_1 at $t_0 + t_1$, C_2 at $t_0 + t_2$, ..., and C_n at $t_0 + t_n$ if the insured is still alive at that time. (For example, consider a life annuity.) Then a payment is made at t^* if t^* equals one of t_i's ($i = 1,\ldots,n$) and t^* is less than t. So we have

$$C(t^*,t) = \begin{cases} C_i & \text{if } t^* = t_i \text{ and } t^* < t,\ i = 1,2,\ldots,n \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

If the payment of the survival benefit is continuous and the rate of payment per annum at time $t_0 + t^*$ is given by the function $\rho(t^*)$, we get

$$\rho(t^*,t) = \begin{cases} \rho(t^*) & \text{if } t^* < t \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Next, let us consider life insurances, whose benefits are payable on death. We will study two important types of them. The first one pays the death benefit at the end of the year of death, and the second one at the moment of death. Note that the year of death is defined as the time period between $t_0 + n - 1$, and $t_0 + n$ if death occurs between the ages of $x + n - 1$ and $x + n$, where n is a positive integer. Furthermore, if the benefit is said to be payable at the moment of death, this does not have to be taken literally. It may take a couple of days before a death can be reported to the insurance company. For example, a death certificate may have to be obtained first, which delays the reporting process. However, a delay of just a few days does not have a significant effect on the finances of an insurance company, so in our computations we can treat the death benefit as being paid at the moment of death.

If a death benefit of C is paid at the moment of death, we get

$$C(t^*,t) = \begin{cases} C & \text{if } t^* = t \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

and if the death benefit is paid at the end of the year of death, we obtain

$$C(t^*,t) = \begin{cases} C & \text{if } t^* = [t] + 1 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The amount C is also called the sum insured.

Still assuming we know the time of death, we can ask what the present

value of the cash flow at time t_0 is. Let us denote it by $g(t)$.

If the cash flow is defined by (1), we get

$$g(t) = \sum_{i=1}^{n} C_i \, v^{t_i} \, I(t_i < t), \tag{5}$$

where $I(t_i < t)$ is the indicator function of the event $t_i < t$.

If the cash flow is given by (2), we obtain

$$g(t) = \int_0^{\infty} \rho(s) \, v^s \, I(s < t) ds. \tag{6}$$

For death benefits we get the following results. If the cash flow is expressed by (3), we have

$$g(t) = C \, v^t \tag{7}$$

and if the cash flow is given by (4), we get

$$g(t) = C \, v^{[t]+1}. \tag{8}$$

If the present value depends on t only through $[t]$, we can rewrite $g(t)$ as $h([t])$. For example, (8) can also be expressed as

$$h(k) = C \, v^{k+1}, \tag{9}$$

where $k = [t]$.

EXAMPLE 1.1. An insurance issued to a life aged 50 makes a payment of \$100 at the age of 51, \$400 at the age of 53, and \$1000 at the age of 55 if the insured is alive at the respective ages. Express the cash flow in the form of $C(t^*,t)$ and the present value of the cash flow at the commencement of the insurance in the form of $g(t)$.

Moreover, determine the cash flow and the present value of the cash flow at the commencement of the insurance, if the insured dies at the age of

a) 54
b) 60.

For the valuation of the cash flow, use a 5% annual rate of interest.

Solution: The cash flow can be expressed as

$$C(t^*,t) = \begin{cases} 100 & \text{if } t^* = 1 \text{ and } t^* < t \\ 400 & \text{if } t^* = 3 \text{ and } t^* < t \\ 1000 & \text{if } t^* = 5 \text{ and } t^* < t \\ 0 & \text{otherwise.} \end{cases}$$

The present value of the cash flow at the commencement of the insurance is

$$g(t) = 100\frac{1}{1.05}I(1 < t) + 400\frac{1}{(1.05)^3}I(3 < t) + 1000\frac{1}{(1.05)^5}I(5 < t).$$

a) If the insured dies at age 54, the cash flow consists of a payment of \$100 at age 51 and \$400 at age 53. The present value is

$$\begin{aligned} g(4) &= 100\frac{1}{1.05} + 400\frac{1}{(1.05)^3} \\ &= \$440.77. \end{aligned}$$

b) If the insured dies at age 60, the cash flow consists of a payment of \$100 at age 51, \$400 at age 53, and \$1000 at age 55. The present value is

$$\begin{aligned} g(10) &= 100\frac{1}{1.05} + 400\frac{1}{(1.05)^3} + 1000\frac{1}{(1.05)^5} \\ &= \$1224.30. \end{aligned}$$

EXAMPLE 1.2. An insurance issued to a life aged 40 makes a continuous payment at an annual rate of \$6000 for 5 years while the insured is alive. Express the cash flow in the form of $\rho(t^*,t)$. Also determine the cash flow, if the insured dies at age

a) 43
b) 53.

Solution: The cash flow can be expressed as

$$\rho(t^*,t) = \begin{cases} 6000 & \text{if } t^* \le 5 \text{ and } t^* < t \\ 0 & \text{otherwise.} \end{cases}$$

a) If the insured dies at age 43, the cash flow consists of a 3 year continuous payment at an annual rate of \$6000.
b) If the insured dies at age 55, the cash flow consists of a 5 year continuous payment at an annual rate of \$6000.

EXAMPLE 1.3. An insurance issued to a life aged 30 provides a death

benefit of \$8000. Based on a 4% annual rate of interest, express the present value of the cash flow in the form of $g(t)$ if the death benefit is payable

a) at the moment of death.
b) at the end of the year of death.

Evaluate these present values if the insured dies at age 54.5.

Solution: a) If the death benefit is payable at the moment of death, the present value is

$$g(t) = 8000 \frac{1}{(1.04)^t}.$$

In particular, if death occurs at age 54.5, the present value is

$$g(24.5) = 8000 \frac{1}{(1.04)^{24.5}} = \$3060.36.$$

b) If the death benefit is payable at the end of the year of death, the present value is

$$g(t) = 8000 \frac{1}{(1.04)^{[t]+1}}.$$

If the insured dies at the age of 54.5, the present value is

$$\begin{aligned} g(24.5) &= 8000 \frac{1}{(1.04)^{[24.5]+1}} \\ &= 8000 \frac{1}{(1.04)^{25}} \\ &= \$3000.93. \end{aligned}$$

The formulas given so far can be used if we know the exact time of death. However, when an insurance starts, it is not known yet how long the insured will live. So we have to use the future lifetime random variable T_x instead of the exact future lifetime. Therefore, the payments of the cash flow can be expressed by $C(t^*,T_x)$ and $\rho(t^*,T_x)$. That means, the payment at any time t^* depends on the random variable T_x. Hence, we get a stochastic cash flow. The reader familiar with advanced probability theory may notice that we are dealing with a stochastic process here. We will use the notation CF for a stochastic cash flow. Although this is the same notation as the one used for non-stochastic cash flows in Chapter 1, it will always be clear from the context what it is supposed to mean.

The present value of a stochastic cash flow at time t_0 can be expressed as $g(T_x)$. This is a random variable whose expected value is denoted by $EPV(CF)$ or EPV and which can be computed as

$$
\begin{aligned}
EPV(CF) = EPV &= E(g(T_x)) \\
&= \int_0^\infty g(t)\, f_x(t)dt \\
&= \int_0^\infty g(t)\, {}_tp_x\, \mu_{x+t} dt,
\end{aligned}
$$

where we used (49) of Section 2.2. If we can write $g(t)$ as $h([t])$, then the expected present value can also be expressed in terms of K_x, since $K_x = [T_x]$:

$$
\begin{aligned}
EPV(CF) = EPV &= E(h(K_x)) \\
&= \sum_{i=0}^{\infty} h(k)\ {}_{k|}q_x.
\end{aligned}
$$

Note that we often refer to the expected present value of the cash flow of an insurance as the present value of the insurance. Moreover, if we do not specify the time at which the present value is to be calculated, we will always mean the time of the commencement of the insurance.

Let us calculate the expected present values for the cash flows defined by (1) to (4).

If the cash flow is defined by (1) then (5) implies

$$
\begin{aligned}
EPV &= E(g(T_x)) \\
&= \int_0^\infty \sum_{i=1}^{n} C_i\, v^{t_i}\, I(t_i < t)\, f_x(t)dt \\
&= \sum_{i=1}^{n} C_i\, v^{t_i} \int_0^\infty I(t_i < t)\, f_x(t)dt \\
&= \sum_{i=1}^{n} C_i\, v^{t_i} \int_{t_i}^\infty f_x(t)dt \\
&= \sum_{i=1}^{n} C_i\, v^{t_i}\, P(T_x \geq t_i) \\
&= \sum_{i=1}^{n} C_i\, v^{t_i}\ {}_{t_i}p_x\,. \qquad (10)
\end{aligned}
$$

The expression $v^t\ {}_tp_x$ appears very often in life insurance. Therefore, we have a special notation for it:

$$
{}_tE_x = v^t\ {}_tp_x.
$$

For example, using Appendix 2, based on a 5% annual rate of interest, we get

$$
\begin{aligned}
{}_1E_{40} = v\, p_{40} &= v(1 - q_{40}) \\
&= \frac{1}{1.05}\,(1 - 0.0027812) \\
&= 0.9497322
\end{aligned}
$$

and

$$
\begin{aligned}
{}_{10}E_{40} = v^{10}\,{}_{10}p_{40} &= v^{10}\,\frac{\ell_{50}}{\ell_{40}} \\
&= \frac{1}{(1.05)^{10}}\,\frac{89509.00}{93131.64} \\
&= 0.5900332.
\end{aligned}
$$

If the cash flow is given by (2), then from (6) we get

$$
\begin{aligned}
EPV &= E(g(T_x)) \\
&= \int_0^\infty \int_0^\infty \rho(s)\, v^s\, I(s < t)\, f_x(t)ds\, dt \\
&= \int_0^\infty \rho(s)\, v^s \left(\int_0^\infty I(s < t)\, f_x(t)dt\right)ds \\
&= \int_0^\infty \rho(s)\, v^s\, {}_sp_x\, ds. \qquad (11)
\end{aligned}
$$

If the cash flow is expressed by (3), then using (7) we obtain

$$
\begin{aligned}
EPV &= E(g(T_x)) \\
&= \int_0^\infty C\, v^t\, f_x(t)dt \\
&= C\int_0^\infty v^t\, {}_tp_x\, \mu_{x+t}\, dt. \qquad (12)
\end{aligned}
$$

If the cash flow is given by (4), then (8) implies

$$
\begin{aligned}
EPV &= E(g(T_x)) \\
&= \int_0^\infty C\, v^{[t]+1}\, f_x(t)dt
\end{aligned}
$$

$$= C \int_0^{\infty} v^{[t]+1} \; {}_t p_x \, \mu_{x+t} \, dt, \tag{13}$$

or using (9) we get

$$\begin{aligned} EPV &= E(h(K_x)) \\ &= \sum_{i=0}^{\infty} C \, v^{k+1} \; {}_{k|}q_x \, . \end{aligned}$$

EXAMPLE 1.4. An insurance issued to a life aged 45 pays survival benefits of \$2000 at the age of 47, \$1000 at the age of 50, and \$500 at the age of 55. On the basis of the mortality table in Appendix 2 and a 3% annual rate of interest, find the present value of the insurance at the commencement of the policy.

Solution: The present value of the insurance is

$$EPV = 2000 \, v^2 \; {}_2p_{45} + 1000 \, v^5 \; {}_5p_{45} + 500 \, v^{10} \; {}_{10}p_{45}$$

$$= \frac{2000 \, v^2 \; \ell_{47} + 1000 \, v^5 \; \ell_{50} + 500 \, v^{10} \; \ell_{55}}{\ell_{45}}$$

$$= \frac{2000 \frac{1}{(1.03)^2} 90880.48 + 1000 \frac{1}{(1.03)^5} 89509.00 + 500 \frac{1}{(1.03)^{10}} 86408.60}{91640.50}$$

$= \$3062.91.$

EXAMPLE 1.5. An insurance issued to a life aged 53 pays a death benefit at the end of the year of death, if death occurs within two years. The amount of the death benefit is \$800 in the first year and \$1200 in the second year. Determine the present value of the insurance at the commencement of the policy based on the mortality table in Appendix 2 and a 4% annual interest rate.

Solution: The present value of the insurance is

$$\begin{aligned} EPV &= 800v \, q_{53} + 1200v^2 \; {}_{1|}q_{53} \\ &= 800v \, q_{53} + 1200v^2 \, p_{53} \, q_{54} \\ &= 800 \, \frac{1}{1.04} \, 0.0075755 + 1200 \, \frac{1}{(1.04)^2} \, (1 - 0.0075755) \, 0.0082364 \\ &= \$14.90. \end{aligned}$$

The expected present value of the stochastic cash flow of an insurance can be interpreted as the average present value of the cash flows of policies issued to many people of the same age. This is why the expected present value of the benefits is usually taken as the single premium payable at the

beginning of the insurance. However, in some special cases, when the portfolio is very small, we may be interested in how much the present value can deviate from the expected present value. In order to do this, we have to find the variance of the present value of the cash flow. It is denoted by $VPV(CF)$ or VPV and it can be determined as

$$\begin{aligned} VPV(CF) &= VPV = V(g(T_x)) \\ &= E(g^2(T_x)) - E^2(g(T_x)) \\ &= \int_0^{\infty} g^2(t)\, f_x(t)dt - \left(\int_0^{\infty} g(t) f_x(t)dt \right)^2 \end{aligned}$$

and if $g(t) = h([t])$, we get

$$\begin{aligned} VPV(CF) &= VPV = V(h(K_x)) \\ &= E(h^2(K_x)) - E^2(h(K_x)) \\ &= \sum_{i=0}^{\infty} h^2(k)\ {}_{k|}q_x - \left(\sum_{i=0}^{\infty} h(k)\ {}_{k|}q_x \right)^2 \end{aligned}$$

Instead of using the expression "variance of the present value of the insurance", we simply say the variance of the insurance. The standard deviation of the insurance, which is the square root of the variance, is a good indicator of how much the present value can fluctuate around the expected value.

The premium calculation will be discussed in Chapter 4 in more detail.

Now consider a life insurance whose benefit is payable if the insured dies after a certain age. For example, payment is made if death occurs after the age of $x + m$; that is, if $T_x > m$. This insurance is called a deferred life insurance.

Next, take a pension whose benefit payments start some time (say m years) after the premium payments begin. Once the insured retires at age $x + m$, the premium payments form a life annuity. However, if we want to determine the premium for this insurance, we have to find the present value m years before the start of the payments. So, we are dealing with a deferred life annuity.

These insurances can be studied in the following general setting.

Consider an insurance issued to a life aged x whose payments start after the age of $x + m$. If death occurs before that age, no payments are made. In other words,

$$C(t^*,t) = 0 \text{ and } \rho(t^*,t) = 0 \text{ if } t^* \leq m \text{ or } t \leq m.$$

This is called an insurance deferred for m years. So, under the condition the insured survives to age $x + m$, the benefit payments can be treated as the

payments of another insurance taken out at time $t_0 + m$ at the age of $x + m$. Let us call the latter a non-deferred insurance. Our goal is to study the relationship between the deferred and the non-deferred insurances.

Since the benefits of the deferred and the non-deferred insurances are the same after the age of $x + m$, we get

$$g_{t_0}(m + t) = v^m\, g_{t_0+m}(t), \quad t \geq 0, \tag{14}$$

where $g_{t_0}(m + t)$ denotes the present value of the cash flow at time t_0 and $g_{t_0+m}(t)$ denotes the present value of the same cash flow at time $t_0 + m$.

We can find the relationship between the probability density functions of T_x and T_{x+m} using (49) of Section 2.2. We get

$$f_x(t) = {}_tp_x\, \mu_{x+t} \tag{15}$$

and

$$f_{x+m}(t) = {}_tp_{x+m}\, \mu_{x+m+t}. \tag{16}$$

Moreover, from (15), we get

$$f_x(t + m) = {}_{m+t}p_x\, \mu_{x+m+t}\,. \tag{17}$$

Using (23) of Section 2.2, we obtain

$${}_mp_x\, {}_tp_{x+m} = {}_{m+t}p_x. \tag{18}$$

So (16), (17), and (18) imply

$$f_x(t + m) = {}_mp_x\, {}_tp_{x+m}\, \mu_{x+m+t} = {}_mp_x\, f_{x+m}(t). \tag{19}$$

Using (14) and (19), we can obtain a relationship between the expected present values at t_0 and $t_0 + m$.

$$\begin{aligned} EPV_{t_0}(CF) &= E(g_{t_0}(T_x)) \\ &= \int_0^\infty g_{t_0}(m + t) f_x(m + t)\,dt \\ &= \int_0^\infty v^m\, g_{t_0+m}(t)\, {}_mp_x\, f_{x+m}(t)\,dt \\ &= v^m\, {}_mp_x \int_0^\infty g_{t_0+m}(t)\, f_{x+m}(t)\,dt \\ &= {}_mE_x\, E(g_{t_0+m}(T_{x+m})) \end{aligned}$$

$$= {}_mE_x\, EPV_{t_0+m}(CF). \tag{20}$$

Next, we want to find the relationship between the variances of the present values at t_0 and at $t_0 + m$. To do this, we have to find the expected values of the squares of the present values first.

$$\begin{aligned}
E(g_{t_0}^2(T_x)) &= \int_0^\infty g_{t_0}^2(m + t)\, f_x(m + t)dt \\
&= \int_0^\infty v^{2m}\, g_{t_0+m}^2(t)\, {}_mp_x\, f_{x+m}(t)dt \\
&= v^{2m}\, {}_mp_x \int_0^\infty g_{t_0+m}^2(t)\, f_{x+m}(t)dt \\
&= v^m\, {}_mE_x\, E(g_{t_0+m}^2(T_{x+m})).
\end{aligned} \tag{21}$$

Thus,

$$VPV_{t_0}(CF) = v^m\, {}_mE_x\, E(g_{t_0+m}^2(T_{x+m})) - {}_mE_x^2\, E^2(g_{t_0+m}(T_{x+m})), \tag{22}$$

which can also be expressed as

$$\begin{aligned}
VPV_{t_0}(CF) = {} & v^m\, {}_mE_x(E(g_{t_0+m}^2(T_{x+m})) - E^2(g_{t_0+m}(T_{x+m}))) \\
& + (v^m\, {}_mE_x - {}_mE_x^2)\, E^2(g_{t_0+m}(T_{x+m})).
\end{aligned} \tag{23}$$

Therefore,

$$VPV_{t_0}(CF) = v^m\, {}_mE_x\, VPV_{t_0+m}(CF) + {}_mE_x(v^m - {}_mE_x)EPV_{t_0+m}^2(CF). \tag{24}$$

Let us examine what happens if we combine two insurances into one. For example, an insurance that pays an amount regularly while the insured is alive and an additional benefit on death can be treated as a combination of a life annuity and a life insurance.

In general, if CF_1 denotes the cash flow corresponding to insurance I_1, and CF_2 is the cash flow corresponding to insurance I_2, the cash flow CF of the combined insurance I_0 consists of all payments of CF_1 and CF_2. We can express this as $CF = CF_1 + CF_2$.

Moreover, if $g_1(t)$ gives the present value of CF_1 at t_0 and $g_2(t)$ is the present value of CF_2 at t_0, then the present value of CF at t_0 is

$$g(t) = g_1(t) + g_2(t). \tag{25}$$

Then we get

$$E(g(T_x)) = E(g_1(T_x)) + E(g_2(T_x)), \tag{26}$$

that is,

$$EPV(CF) = EPV(CF_1) + EPV(CF_2). \tag{27}$$

Unfortunately, $VPV(CF)$ usually cannot be obtained in a simple way from CF_1 and CF_2. In fact, squaring both sides of (25), we obtain

$$g^2(t) = g_1^2(t) + g_2^2(t) + 2g_1(t)\, g_2(t)$$

and so

$$E(g^2(T_x)) = E(g_1^2(T_x)) + E(g_2^2(T_x)) + 2E(g_1(T_x)\, g_2(T_x)).$$

Since $g_1(T_x)$ and $g_2(T_x)$ are usually not independent, evaluating $E(g_1(T_x)\, g_2(T_x))$ can be cumbersome. However, there are certain cases when the function $g_1(t)\, g_2(t)$ is identically zero. This happens, if for every $T_x = t$, either insurance I_1 or insurance I_2 does not pay any benefits. For example, insurance I_1 pays a benefit at the moment of death if death occurs in the first n years, and insurance I_2 is a life annuity deferred for n years. In general, $g_1(t)\, g_2(t)$ is identically zero, if there exists an n, such that insurance I_1 pays only if $T_x < n$ and insurance I_2 pays only if $T_x \geq n$. Then,

$$g_1(t) = 0 \text{ if } t \leq n,$$

and

$$g_2(t) = 0 \text{ if } t > n.$$

In any case, if $g_1(t)\, g_2(t) = 0$ for every t, we have

$$E(g_1(T_x)\, g_2(T_x)) = 0.$$

Thus,

$$E(g^2(T_x)) = E(g_1^2(T_x)) + E(g_2^2(T_x)). \tag{28}$$

Then, using (26) and (28), we obtain

$$\begin{aligned} VPV(CF) &= V(g(T_x)) \\ &= E(g_1^2(T_x)) + E(g_2^2(T_x)) - (E(g_1(T_x)) + E(g_2(T_x)))^2 \end{aligned}$$

$$= VPV(CF_1) + EPV^2(CF_1) + VPV(CF_2) + EPV^2(CF_2) - (EPV(CF_1) + EPV(CF_2))^2$$
$$= VPV(CF_1) + VPV(CF_2) - 2EPV(CF_1)\, EPV(CF_2). \tag{29}$$

From (26) and (28), we can also obtain

$$\begin{aligned} VPV(CF_2) &= V(g_2(T_x)) \\ &= E(g^2(T_x)) - E(g_1^2(T_x)) - (E(g(T_x) - E(g_1(T_x)))^2 \\ &= VPV(CF) + EPV^2(CF) - (VPV(CF_1) + EPV^2(CF_1)) \\ &\quad - (EPV(CF) - EPV(CF_1))^2. \end{aligned} \tag{30}$$

Formula (30) is especially useful, if we have already derived the expected values and variances of CF and CF_1 and we are interested in finding those of CF_2.

Next, consider the following arrangement. A person aged x at t_0 agrees to pay given amounts of money to the insurance company at certain times if he/she is still alive then. For example, he/she pays an amount of C_1 at time $t_0 + t_1$, C_2 at $t_0 + t_2,\ldots,$ and C_n at $t_0 + t_n$. In return, the insurance company promises to pay the insured a sum of C at $t_0 + t_e$ (where $t_e \geq t_n$) if he/she survives to that time. Assuming $C_1, C_2,\ldots,C_n$ are given, what should C be? It is reasonable to choose a C such that the expected present value at t_0 of the payments made by the insured equals the expected present value at t_0 of the payment made by the insurance company:

$$\sum_{j=1}^{n} C_j\, v^{t_j}\, {}_{t_j}p_x = C\, v^{t_e}\, {}_{t_e}p_x. \tag{31}$$

So we can express C from (31) as

$$C = \frac{1}{v^{t_e}\, {}_{t_e}p_x} \sum_{j=1}^{n} C_j\, v^{t_j}\, {}_{t_j}p_x = \frac{EPV_{t_0}(CF)}{{}_{t_e}E_x}. \tag{32}$$

Since C is the sum the insurance company promises to pay after receiving the payments of the insured, it is called the accumulation of the stochastic cash flow. Note that it is different from the accumulation introduced in Chapter 1. Using the accumulation concept of financial mathematics, we would get

$$\begin{aligned} A &= \sum_{j=1}^{n} C_j (1+i)^{(t_e - t_j)} \\ &= \sum_{j=1}^{n} C_j\, v^{-(t_e - t_j)}, \end{aligned} \tag{33}$$

where i is the annual interest rate. Comparing (32) and (33), we find

$$C = \sum_{j=1}^{n} C_j v^{-(t_e - t_j)} \frac{{}_{t_j}p_x}{{}_{t_e}p_x}$$
$$= \sum_{j=1}^{n} C_j v^{-(t_e - t_j)} \frac{1}{{}_{t_e - t_j}p_{x+t_j}} > A$$

where the inequality holds, since ${}_{t_e - t_0}p_{x+t_i} < 1$.

What this shows is that if the insured survives to time $t_0 + t_e$, the money paid by him/her does not grow up to C by that time if it is invested at an annual rate of interest of i. On the other hand, if the insured dies before time t_e, no money will be paid by the insurance company, although it has received some. However, if there is a large number of people who have the same insurance, the insurance company can promise to pay an amount of C to the survivors if C is determined from the expected value equation (31).

EXAMPLE 1.6. An insured pays an insurance company $800 at the age of 55, $1000 at the age of 56, and $2000 at the age of 58. How much can the insurance company pay back at the age of 60 on survival? Base the calculations on the mortality table in Appendix 2 and a 4% annual rate of interest.

Solution: The company can pay back the accumulated value of the cash flow at age 60. From (32), we get

$$c = \frac{1}{v^5 \, {}_5p_{55}} \left(800 + 1000v \, {}_1p_{55} + 2000v^3 \, {}_3p_{55}\right)$$
$$= \frac{1}{v^5 \dfrac{\ell_{60}}{\ell_{55}}} \left(800 + 1000v \frac{\ell_{56}}{\ell_{55}} + 2000v^3 \frac{\ell_{58}}{\ell_{55}}\right)$$
$$= \frac{800\,\ell_{55} + 1000v\,\ell_{56} + 2000v^3\,\ell_{58}}{v^5\,\ell_{60}}$$
$$= \frac{800 \times 86408.60 + 1000\frac{1}{1.04}85634.33 + 2000\frac{1}{(1.04)^3}83898.25}{\frac{1}{(1.04)^5}81880.73}$$
$$= \$4467.13.$$

So the company can pay $4467.13 at the age of 60 on survival.

Now, consider a stochastic cash flow consisting of premium payments made by an insured to the insurance company and survival or death benefits

paid by the insurance company to the insured. Assume the stochastic cash flow starts at time t_0 when the insured is aged x and all possible premium and benefit payments take place before time $t_0 + t_e$. We can ask what the company can promise to pay to the insured on survival at time $t_0 + t_e$. This amount is called the accumulated value of the stochastic cash flow. It is denoted by $ACV(CF)$ or ACV. We can determine it by equating the present value at time t_0 of the stochastic cash flow to that of the survival benefit payment at time $t_0 + t_e$:

$$EPV_{t_0}(CF) = {}_{t_e}E_x \, ACV(CF).$$

Thus,

$$ACV = ACV(CF) = \frac{EPV_{t_0}(CF)}{{}_{t_e}E_x}. \tag{34}$$

If it is not clear from the context what t_e is, we can use the notation $ACV_{t_e}(CF)$ or ACV_{t_e}.

Of course, (34) only makes sense if ${}_{t_e}E_x$ is not zero; that is, if $x + t_e < \omega$. However, in practical applications this condition is always satisfied so we will not consider it any more.

We want to emphasize again that this result is only meaningful if we have a large number of policyholders having the same insurance. Then, we can say the accumulated value is the share of one policyholder from the fund built up from all the policies.

EXAMPLE 1.7. An insurance is issued to a person aged 50. The insured pays a premium of \$70 at age 50 and \$120 at age 51. The company pays a death benefit of \$8000 at the end of the year of death, if it occurs between the ages of 51 and 52. Find the amount the company can pay to the insured at age 52 on survival. Use the mortality table in Appendix 2 and a 3% annual rate of interest.

Solution: The amount of the survival benefit can be obtained as the accumulated value of the stochastic cash flow consisting of the premium payments and the death benefit payments:

$$\begin{aligned} ACV &= \frac{70 + 120v \, {}_1p_{50} - 8000v^2 \, {}_{1|}q_{50}}{v^2 \, {}_2p_{50}} \\ &= \frac{70 + 120v \dfrac{\ell_{51}}{\ell_{50}} - 8000v^2 \dfrac{d_{51}}{\ell_{50}}}{v^2 \dfrac{\ell_{52}}{\ell_{50}}} \end{aligned}$$

$$= \frac{70\,\ell_{50} + 120v\;\ell_{51} - 8000v^2\,d_{51}}{v^2\;\ell_{52}}$$

$$= \frac{70 \times 89509.00 + 120\,\frac{1}{1.03}\,88979.11 - 8000\,\frac{1}{(1.03)^2}\,571.4316}{\frac{1}{(1.03)^2}\,88407.68}$$

$$= \$147.88.$$

So, the company can pay a survival benefit of \$147.88 at the age of 52.

Next, consider an insurance issued to a life aged t_0, consisting of premium payments and survival and death benefit payments. Assume the expected present value of the premium payments at time t_0 equals the expected present value of the benefits. So, if we give positive signs to the benefit payments and negative signs to the premium payments, we obtain

$$EPV_{t_0}(CF) = 0,$$

where CF is the cash flow of the insurance.

Let us select a positive t with the property that (premium or benefit) payments after time $t_0 + t$ cannot be made unless the insured survives to time $t_0 + t$.

In other words,

$$C(t^*,t_1) = 0 \text{ and } \rho(t^*,t_1) = 0 \text{ if } t_1 < t < t^*.$$

Note that a t greater than zero does not necessarily have this property. For example, if a death benefit is payable at the end of the year of death and $t = \frac{1}{2}$, then a death benefit will be paid after $t_0 + t$ (namely at $t_0 + 1$) if the insured dies before $t_0 + t$. On the other hand, if $t = 1$, t has the required property.

Let us return to the case when t satisfies the given condition. Let us denote the cash flow of the (premium and benefit) payments before time $t_0 + t$ by CF_1 and the cash flow of the (premium and benefit) payments after time $t_0 + t$ by CF_2. If a payment is due at exactly time $t_0 + t$, we have to assign it to one of the two cash flows. We get $CF = CF_1 + CF_2$. Thus, (27) implies

$$EPV_{t_0}(CF) = EPV_{t_0}(CF_1) + EPV_{t_0}(CF_2).$$

Therefore,

$$EPV_{t_0}(CF_1) + EPV_{t_0}(CF_2) = 0. \tag{35}$$

Using (20), we obtain

$$EPV_{t_0}(CF_2) = {}_tE_x \; EPV_t(CF_2). \tag{36}$$

Furthermore, the payments of CF_1 are made before $t_0 + t$, hence its accumulated value is defined at $t_0 + t$. When we compute the accumulated value, we usually give the premium payments positive signs and the benefit payments negative signs. So, we are interested in the accumulated value $ACV_t(\text{-}CF_1)$. Now, from (34), we obtain

$$ACV_t(\text{-}CF_1) = \frac{EPV_{t_0}(\text{-}CF_1)}{{}_tE_x}. \tag{37}$$

Hence (35), (36), and (37) imply

$$ACV_t(\text{-}CF_1) = EPV_t(CF_2). \tag{38}$$

Equation (38) will play an important role in Chapter 5.

At this point, we are ready to start a systematic study of the different types of insurances.

PROBLEMS

1.1. An insurance issued to a life aged 30 pays a survival benefit of \$500 at the age of 35, \$700 at the age of 40, and \$1000 at the age of 45. Express the cash flow in the form of $C(t^*,t)$ and the present value of the cash flow at the commencement of the insurance in the form of $g(t)$. Determine the present value of the cash flow at the commencement of the insurance, if the insured dies at age

a) 31
b) 42
c) 70.

For the valuation of the cash flow, use a 4% annual interest rate.

1.2. An insurance issued to a life aged 35 makes a continuous payment at a rate of \$2000 per annum while the insured is alive. Express the cash flow in the form of $C(t^*,t)$ and the present value of the cash flow at the commencement of the insurance in the form of $g(t)$. Find the cash flow and determine the present value of the cash flow at the commencement of the insurance, if the insured dies at the age of 40. For the valuation of the cash flow, use a 5% annual rate of interest.

1.3. An insurance issued to a life aged 40 provides a death benefit of \$5000. Express the cash flow in the form of $C(t^*,t)$ and the present value of the cash flow at the commencement of the insurance in the form of $g(t)$ if the death benefit is payable

a) at the moment of death.
b) at the end of the year of death.

Under both of conditions (a) and (b), determine the present value of the cash flow at the commencement of the insurance if the insured dies at the age of 57.2. Use a 4% annual interest rate.

1.4. Based on $i = 5\%$ and the mortality table in Appendix 2, obtain

a) ${}_8E_{50}$.
b) ${}_{10}E_{60}$.

1.5. Obtain the expected present value of the insurance, given in Problem 1.1, at the commencement of the insurance based on the mortality table in Appendix 2.

1.6. An insurance issued to a life aged 30 pays a death benefit of \$2000 at the end of the year of death, if death occurs between the ages of 30 and 31. A death benefit of \$1500 is paid at the end of the year of death, if death occurs between the ages 31 and 32. Moreover, there is a survival benefit of \$1000 payable on survival to age 32. Find the present value of the insurance at the commencement of the policy based on the mortality table in Appendix 2 and a 3% annual rate of interest.

1.7. An insured pays an insurance company \$600 at the age of 40, \$800 at the age of 42, and \$1500 at the age of 46. How much can the insurance company pay the insured at the age of 50 on survival? Use the mortality table in Appendix 2 and a 5% annual rate of interest.

1.8. A person pays \$100 to an insurance company at the age of 40. The company pays a death benefit of \$5000 at the end of the year of death, if death occurs within three years. Determine the amount of the survival benefit the company can offer to pay at the age of 43. Use the mortality table in Appendix 2 and a 4% annual interest rate.

3.2. Pure Endowments

One of the simplest types of insurances is the one which pays an amount of C if the insured survives to a certain age. This is called a pure endowment insurance. Obviously, it is enough to study the case when $C = 1$. Hence we will discuss the following situation.

An insurance issued to a life aged x pays \$1 at age of $x + n$ on survival. Here, n is a fixed positive number.

That means, if $T_x \geq n$, then an amount of \$1 is paid at $t_0 + n$, whose present value at t_0 is v^n. If $T_x < n$, no money is paid. So, if we define

$$g(t) = \begin{cases} v^n & \text{if } t \geq n \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

then $g(T_x)$ is the present value of the benefit at t_0. Its expected value, that is, $E(g(T_x))$ is denoted by $A_{\overset{1}{x:\overline{n}|}}$. The symbol "1" above n in the notation means that the benefit is paid if the n years end sooner than the life of (x). As a result, we have

$$\begin{aligned} A_{\overset{1}{x:\overline{n}|}} &= E(g(T_x)) \\ &= v^n \, P(T_x \geq n) \\ &= v^n \, {}_np_x \\ &= {}_nE_x. \end{aligned} \tag{2}$$

If we want to use (2) to evaluate $A_{\overset{1}{x:\overline{n}|}}$ numerically, we have to find the values of ℓ_x and ℓ_{x+n} from a mortality table and also compute v^n.

This procedure can be shortened if we rewrite ${}_nE_x$ in the following way

$${}_nE_x = \frac{v^{x+n} \, \ell_{x+n}}{v^x \, \ell_x}. \tag{3}$$

Note that if we define the function

$$D_x = v^x \, \ell_x, \text{ for } x \geq 0, \tag{4}$$

we get

$$A_{\overset{1}{x:\overline{n}|}} = {}_nE_x = \frac{D_{x+n}}{D_x}. \tag{5}$$

The function D_x is called a commutation function. Its values for integer x's,

n's, and 6% annual rate of interest are tabulated in Appendix 2.

The following property of ${}_nE_x$ follows from (5) immediately:

$$_{n+m}E_x = {}_nE_{x+m}\ {}_mE_x. \tag{6}$$

Let us derive the variance of the insurance. We can see from (1) that $g^2(t)$ evaluated at the discount factor v is the same as $g(t)$ evaluated at the discount factor v^2. So, denoting the variance of the insurance by $V(A_{x:\overline{n}|}^{\ \ 1})$ we get

$$V(A_{x:\overline{n}|}^{\ \ 1}) = {}^2A_{x:\overline{n}|}^{\ \ 1} - (A_{x:\overline{n}|}^{\ \ 1})^2, \tag{7}$$

where ${}^2A_{x:\overline{n}|}^{\ \ 1}$ (or ${}_n^2E_x$) denotes $A_{x:\overline{n}|}^{\ \ 1}$ evaluated at the discount factor v^2.

Similar notations will be used throughout the chapter.

Note that $A_{x:\overline{n}|}^{\ \ 1}$ (or ${}_nE_x$) evaluated at the discount factor v^2 is

$$\begin{aligned}
{}^2A_{x:\overline{n}|}^{\ \ 1} = {}_n^2E_x &= v^{2n}\,\frac{\ell_{x+n}}{\ell_x} \\
&= v^n\,\frac{v^{x+n}\,\ell_{x+n}}{v^x\,\ell_x} \\
&= v^n\,\frac{D_{x+n}}{D_x} \\
&= v^n\,{}_nE_x.
\end{aligned} \tag{8}$$

Thus,

$$\begin{aligned}
V(A_{x:\overline{n}|}^{\ \ 1}) &= v^n\,{}_nE_x - {}_nE_x^2 \\
&= (v^n - {}_nE_x)\,{}_nE_x \\
&= v^n(1 - {}_np_x){}_nE_x.
\end{aligned} \tag{9}$$

EXAMPLE 2.1. An insurance on a life aged 25 pays a sum of \$5000 on survival to age 60. Find the present value of the insurance based on a 6% annual rate of interest. Obtain the standard deviation of the insurance, as well.

Solution: First, we have to determine $A_{25:\overline{35}|}^{\ \ \ \ 1}$. We can do this in two different ways. We can either write

$$
\begin{aligned}
A^{\;\;1}_{25:\overline{35}|} &= v^{35}\,{}_{35}p_{25} \\
&= v^{35}\,\frac{\ell_{60}}{\ell_{25}} \\
&= 0.13011\,\frac{81880.73}{95650.15} \\
&= 0.13011 \times 0.85604 \\
&= 0.11138
\end{aligned}
$$

or

$$
A^{\;\;1}_{25:\overline{35}|} = \frac{D_{60}}{D_{25}} = \frac{2482.16}{22286.35} = 0.11138.
$$

Hence, the present value is $\$5000 \times 0.11138 = \$556.90$. In order to obtain the standard deviation, we need to find $V(A^{\;\;1}_{25:\overline{35}|})$,

$$
\begin{aligned}
V(A^{\;\;1}_{25:\overline{35}|}) &= v^{35}(1 - {}_{35}p_{25})\,{}_{35}E_{25} \\
&= 0.13011(1 - 0.85604)\,0.11138 \\
&= 0.0020862.
\end{aligned}
$$

As a result, the standard deviation is $\$5000\sqrt{0.0020862} = \$228.37$.

PROBLEMS

2.1. Based on a 4% annual interest rate, evaluate

a) ${}_5E_{40}$
b) D_{35}
c) $A^{\;\;1}_{25:\overline{10}|}$

2.2. Based on a 6% annual rate of interest, find

a) ${}_{10}E_{25}$
b) D_{50}
c) $A^{\;\;1}_{45:\overline{20}|}$

2.3. An insurance issued to a life aged 40 pays \$2000 at the age of 50 on survival. Find the present value and the standard deviation of the insurance, based on a 5% annual rate of interest.

2.4. A 20 year pure endowment insurance of \$1500 is taken out by a person aged 45. Find the present value and the standard deviation of the insurance, using a 6% annual interest rate.

3.3. Life Insurances

In this section, we will study insurances whose benefits are payable on death. They are called life insurances.

First, let us focus on life insurances that pay the death benefit at the end of the year of death. Consider the following general case.

An insured is aged x at the time the insurance is taken out. The insurance pays a benefit of \$1 at the end of the year of death, if death occurs between the ages of $x+m$ and $x+m+n$, where m is a fixed nonnegative and n is a fixed positive integer.

This is an n year temporary (or term) insurance, deferred for m years.

The present value of the insurance is denoted by ${}_{m|}A^{\,1}_{x:\overline{n}|}$ or ${}_{m|n}A_x$. We will use the first notation in the book since it fits better the overall structure of the symbols. The 1 above x means that the benefit is payable if the life ends sooner than the insurance period.

The present value of the cash flow can be expressed as $h(K_x)$, where

$$h(k) = \begin{cases} 0 & \text{if } k < m \\ v^{k+1} & \text{if } m \le k < m+n \\ 0 & \text{if } m+n \le k. \end{cases} \tag{1}$$

Thus,

$$\begin{aligned} {}_{m|}A^{\,1}_{x:\overline{n}|} = E(h(K_x)) &= \sum_{k=m}^{m+n-1} v^{k+1}\, p(K_x = k) \\ &= \sum_{k=m}^{m+n-1} v^{k+1}\, {}_{k|}q_x \\ &= \sum_{k=m}^{m+n-1} v^{k+1} \frac{d_{x+k}}{\ell_x} \\ &= \sum_{k=m}^{m+n-1} \frac{v^{x+k+1}\, d_{x+k}}{v^x\, \ell_x} . \end{aligned}$$

Let us define the commutation functions C_x and M_x by

$$C_x = v^{x+1} d_x \tag{2}$$

and

$$M_x = \sum_{k=0}^{\infty} C_{x+k} \, . \tag{3}$$

Note that since ℓ_x becomes zero if x exceeds ω, C_x is also zero for these x's, so M_x is a finite sum. Moreover, M_x is also zero if x is greater than ω. The values of C_x and M_x at a 6% annual rate of interest are tabulated in Appendix 2.

Using the commutation functions, we get

$$_{m|}A^{1}_{x:\overline{n}|} = \sum_{k=m}^{m+n-1} \frac{C_{x+k}}{D_x} = \frac{M_{x+m} - M_{x+m+n}}{D_x} \, . \tag{4}$$

The variance of the insurance is

$$V(_{m|}A^{1}_{x:\overline{n}|}) = {}^{2}_{m|}A^{1}_{x:\overline{n}|} - (_{m|}A^{1}_{x:\overline{n}|})^2. \tag{5}$$

Note that

$$_{k-1|}A^{1}_{x:\overline{1}|} = \frac{C_{x+k-1}}{D_x},$$

so $\frac{C_{x+k-1}}{D_x}$ is the present value at the age of x of a death benefit of \$1 payable at the end of year k, if death occurs in year k.

If $m = 0$ and n is infinity (or more precisely, n is so large that $x + m + n > \omega$), the insurance is called a whole life insurance and its present value is denoted by A_x. So in this case, the death benefit is paid no matter when the insured dies. From (4) and (5), we get

$$A_x = \frac{M_x}{D_x} \tag{6}$$

and

$$V(A_x) = {}^2A_x - A_x^2. \tag{7}$$

The values of A_x and ${}^2A_x$ at a 6% annual rate of interest are tabulated in

Appendix 2. The present value ${}_{m|}A^{\;1}_{x:\overline{n}|}$ can also be expressed in terms of A_x. In fact, from (4) and (6) we get

$$ {}_{m|}A^{\;1}_{x:\overline{n}|} = {}_mE_x\, A_{x+m} - {}_{m+n}E_x\, A_{x+m+n}. \tag{8}$$

Moreover,

$$ {}^{2}_{m|}A^{\;1}_{x:\overline{n}|} = v^{2m}\ \frac{\ell_{x+m}}{\ell_x}\ {}^2A_{x+m} - v^{2(m+n)}\ \frac{\ell_{x+m+n}}{\ell_x}\ {}^2A_{x+m+n},$$

so

$$ {}^{2}_{m|}A^{\;1}_{x:\overline{n}|} = v^m\ {}_mE_x\ {}^2A_{x+m} - v^{m+n}\ {}_{m+n}E_x\ {}^2A_{x+m+n}.$$

If $m = 0$ and n is not infinity, we use the notation $A^{\;1}_{x:\overline{n}|}$. Thus,

$$ A^{\;1}_{x:\overline{n}|} = \frac{M_x - M_{x+n}}{D_x} \tag{9}$$

and

$$ V(A^{\;1}_{x:\overline{n}|}) = {}^2A^{\;2}_{x:\overline{n}|} - (A^{\;1}_{x:\overline{n}|})^2. \tag{10}$$

We can write

$$ {}^2A^{\;1}_{x:\overline{n}|} = {}^2A_x - v^n\ {}_nE_x\ {}^2A_{x+n}.$$

If $m > 0$ and n is infinity, we use the notation ${}_{m|}A_x$. We get

$$ {}_{m|}A_x = \frac{M_{x+m}}{D_x} \tag{11}$$

and

$$ V({}_{m|}A_x) = {}^{2}_{m|}A_x - ({}_{m|}A_x)^2, \tag{12}$$

where we have

$$ {}^{2}_{m|}A_x = v^m\ {}_mE_x\ {}^2A_{x+m}.$$

Next, let us study the case when the death benefit varies with time. An

important type of varying insurances is defined in the following way.

An insured is aged x at the time the insurance is taken out. The insurance pays $\$k$ at the end of year k, if death occurs in that year, and $k < n$. The number n is a fixed positive integer.

The present value of the insurance is denoted by $(IA)^{1}_{x:\overline{n}|}$.

The present value of the cash flow can be expressed as $h(K_x)$, where

$$h(k) = \begin{cases} k\,v^{k+1} & \text{if } k < n \\ 0 & \text{if } k \geq n. \end{cases} \tag{13}$$

Note that this varying insurance can be expressed as the sum of a non-deferred n year term insurance, an $n - 1$ year insurance deferred for 1 year, an $n - 2$ year insurance deferred for 2 years, ..., and a 1 year insurance deferred for $n - 1$ years.

So, using (27) of Section 3.1, repeatedly, we obtain

$$(IA)^{1}_{x:\overline{n}|} = \sum_{m=0}^{n-1} {}_{m|}A^{1}_{x:\overline{n-m}|} = \sum_{m=0}^{n-1} \frac{M_{x+m} - M_{x+n}}{D_x}.$$

We define the commutation function R_x by

$$R_x = \sum_{k=0}^{\infty} M_{x+k}. \tag{14}$$

The values of R_x at a 6% annual rate of interest are also tabulated in Appendix 2. Thus, we can write

$$(IA)^{1}_{x:\overline{n}|} = \frac{R_x - R_{x+n} - n\,M_{x+n}}{D_x}. \tag{15}$$

Unfortunately, $g^2(k)$ is not the same as $g(k)$ evaluated at the discount factor v^2. The reader may try to derive the variance of the insurance directly from the definition

$$V(h(K_x)) = E(h^2(K_x)) - (E(h(K_x))^2.$$

If n is infinity, from (15) we get

$$(IA)_x = \frac{R_x}{D_x}. \tag{16}$$

Thus, we can write

$$(IA)^{1}_{x:\overline{n}|} = (IA)_x - {}_nE_x(IA)_{x+n} - n\;{}_nE_x\,A_{x+n}. \tag{17}$$

EXAMPLE 3.1. A person aged 50 takes out a whole life insurance that pays an amount of \$1500 at the end of the year of death. Using a 6% annual rate of interest, find the present value of the insurance. Also, find the standard deviation of the insurance.

Solution: First, we determine A_{50}. We can either do it by computing

$$A_{50} = \frac{M_{50}}{D_{50}} = \frac{1210.1957}{4859.30} = 0.2490473$$

or directly looking up A_{50} in the table

$$A_{50} = \frac{249.0475}{1000} = 0.2490475.$$

Thus, the present value is $\$1500 \times 0.2490475 = \$373.57$. Next, we find $V(A_{50})$.

$$V(A_{50}) = {}^2A_{50} - A_{50}^2 = 0.0947561 - (0.2490473)^2 = 0.0327315.$$

Therefore, the standard deviation is $\$1500\sqrt{0.0327315} = \$271.38$.

EXAMPLE 3.2. A temporary insurance is taken out by a person aged 40. The insurance pays a benefit of \$2000 at the end of the year of death if death occurs before the age of 60. Using a 6% annual rate of interest, find the present value of the insurance. Also, find the standard deviation of the insurance.

Solution: First, we need to find $A^{1}_{40:\overline{20}|}$. We have

$$A^{1}_{40:\overline{20}|} = \frac{M_{40} - M_{60}}{D_{40}} = \frac{1460.7038 - 916.2423}{9054.46} = 0.0601318.$$

Hence, the present value is $\$2000 \times 0.0601318 = \$120.26$. Next, we determine $V(A^{1}_{40:\overline{20}|})$. Since

$$V(A^{1}_{40:\overline{20}|}) = {}^2A^{1}_{40:\overline{20}|} - (A^{1}_{40:\overline{20}|})^2$$

and

$${}^2A^{1}_{40:\overline{20}|} = {}^2A_{40} - {}^{2}_{20}E_{40}\;{}^2A_{60} = {}^2A_{40} - v^{20}\,\frac{D_{60}}{D_{40}}\,{}^2A_{60}$$

$$= 0.0486332 - 0.31180 \frac{2482.16}{9054.46} 0.1774113 = 0.0334688,$$

we obtain

$$V(A^{\;\;1}_{40:\overline{20}|}) = 0.0334688 - (0.0601318)^2 = 0.0298530.$$

So, the standard deviation is $\$2000 \sqrt{0.0298530} = \$345.56$.

EXAMPLE 3.3. A varying whole life insurance is taken out at the age of 55. The death benefit is \$2000 at the end of the first year and increases by \$100 each year. Find the present value of the insurance at a 6% annual rate of interest.

Solution: The present value of the insurance can be expressed as

$$100(IA)_{55} + (2000 - 100)\, A_{55}.$$

Now, we have

$$(IA)_{55} = \frac{R_{55}}{D_{55}} = \frac{18505.9227}{3505.37} = 5.2793065$$

and

$$A_{55} = \frac{M_{55}}{D_{55}} = \frac{1069.6405}{3505.37} = 0.3051440.$$

Thus, the present value is $100 \times 5.2793065 + 1900 \times 0.3051440 = \$1107.70$.

Next, we examine life insurances that pay the death benefit at the moment of death. Again, we will consider a general setting first.

An insured is aged x at the time the insurance is taken out. The insurance pays a benefit of \$1 at the moment of death, if death occurs between the ages of $x + m$ and $x + m + n$, where m is a fixed nonnegative and n is a fixed positive integer.

This is an n year temporary (or term insurance), deferred for m years. The present value of the insurance is denoted by ${}_{m|}\overline{A}^{\;1}_{x:\overline{n}|}$ or ${}_{m|n}\overline{A}_x$. We will use the first notation in the book. The present value of the cash flow is $g(T_x)$, where

$$g(t) = \begin{cases} 0 & \text{if } t < m \\ v^t & \text{if } m \le t < m + n \\ 0 & \text{if } m + n \le t. \end{cases} \tag{18}$$

Hence,

$$
\begin{aligned}
{}_{m|}\bar{A}^{1}_{x:\overline{n}|} = E(g(T_x)) &= \int_m^n v^t f_x(t)\,dt \\
&= \sum_{k=m}^{n-1} \int_k^{k+1} v^t\, {}_tp_x\, \mu_{x+t}\,dt \\
&= \sum_{k=m}^{n-1} \int_k^{k+1} \frac{v^{x+t}\,\ell_{x+t}}{v^x\,\ell_x}\,\mu_{x+t}\,dt \\
&= \sum_{k=m}^{n-1} \frac{\int_0^1 v^{x+k+z}\,\ell_{x+k+z}\,\mu_{x+k+z}\,dz}{v^x\,\ell_x}.
\end{aligned}
$$

We define the commutation function

$$\bar{C}_x = \int_0^1 v^{x+t}\,\ell_{x+t}\,\mu_{x+t}\,dt, \tag{19}$$

which can also be written as

$$\bar{C}_x = \int_0^1 D_{x+t}\,\mu_{x+t}\,dt. \tag{20}$$

Furthermore, we introduce the commutation function $\bar{M}_x$

$$\bar{M}_x = \sum_{k=0}^{\infty} \bar{C}_{x+k}. \tag{21}$$

Thus,

$${}_{m|}\bar{A}^{1}_{x:\overline{n}|} = \sum_{k=m}^{m+n-1} \frac{\bar{C}_{x+k}}{D_x} = \frac{\bar{M}_{x+m} - \bar{M}_{x+m+n}}{D_x}. \tag{22}$$

The variance of the insurance is

$$V({}_{m|}\bar{A}^{\,1}_{x:\overline{n}|}) = {}^{2}_{m|}\bar{A}^{\,1}_{x:\overline{n}|} - ({}_{m|}\bar{A}^{\,1}_{x:\overline{n}|})^2. \tag{23}$$

Since mortality tables often do not contain $\bar{C}_x$ and $\bar{M}_x$, we have to use an approximation. Considering the integral in (19), we see that the exponent of v varies between x and $x + 1$. Hence, we can get a reasonable approximation if we replace v^{x+t} by $v^{x+\frac{1}{2}}$ throughout the integration interval:

$$\bar{C}_x \approx v^{x+\frac{1}{2}} \int_0^1 \ell_{x+t}\,\mu_{x+t}\,dt.$$

From (15) of Section 2.2, we get

$$\int_0^1 \ell_{x+t}\,\mu_{x+t}\,dt = d_x.$$

Thus,

$$\bar{C}_x \approx v^{x+\frac{1}{2}} d_x = (1+i)^{\frac{1}{2}} v^{x+1} d_x,$$

so we get the approximations

$$\bar{C}_x \approx (1+i)^{\frac{1}{2}} C_x, \tag{24}$$

$$\bar{M}_x \approx (1+i)^{\frac{1}{2}} M_x, \tag{25}$$

and

$${}_{m|}\bar{A}^{\,1}_{x:\overline{n}|} \approx (1+i)^{\frac{1}{2}}\, {}_{m|}A^{1}_{x:\overline{n}|}, \tag{26}$$

or equivalently,

$${}_{m|}\bar{A}^{\,1}_{x:\overline{n}|} \approx v^{-\frac{1}{2}}\, {}_{m|}A^{1}_{x:\overline{n}|}.$$

Therefore, we also get

$$
\begin{aligned}
{}^{2}_{m|}\bar{A}^{1}_{x:\overline{n}|} &\approx v^{-1}\,{}^{2}_{m|}A^{1}_{x:\overline{n}|} \\
&= (1+i)\,{}^{2}_{m|}A^{1}_{x:\overline{n}|}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
V({}_{m|}\bar{A}^{1}_{x:\overline{n}|}) &\approx (1+i)\,{}^{2}_{m|}A^{1}_{x:\overline{n}|} - ((1+i)^{\frac{1}{2}}\,{}_{m|}A^{1}_{x:\overline{n}|})^2 \\
&= (1+i)({}^{2}_{m|}A^{1}_{x:\overline{n}|} - ({}_{m|}A^{1}_{x:\overline{n}|})^2) \\
&= (1+i)\ V({}_{m|}A^{1}_{x:\overline{n}|}).
\end{aligned}
\tag{27}
$$

Formulas similar to (6) through (12) can also be obtained

$$
\bar{A}_x = \frac{\bar{M}_x}{D_x}, \tag{28}
$$

$$
V(\bar{A}_x) = {}^{2}\bar{A}_x - \bar{A}_x^{2}, \tag{29}
$$

$$
{}_{m|}\bar{A}^{1}_{x:\overline{n}|} = {}_{m}E_x\ \bar{A}_{x+m} - {}_{m+n}E_x\,\bar{A}_{x+m+n}, \tag{30}
$$

$$
\bar{A}^{1}_{x:\overline{n}|} = \frac{\bar{M}_x - \bar{M}_{x+n}}{D_x}, \tag{31}
$$

$$
V(\bar{A}^{1}_{x:\overline{n}|}) = {}^{2}\bar{A}^{1}_{x:\overline{n}|} - (\bar{A}^{1}_{x:\overline{n}|})^2, \tag{32}
$$

$$
{}_{m|}\bar{A}_x = \frac{\bar{M}_{x+m}}{D_x}, \tag{33}
$$

and

$$
V({}_{m|}\bar{A}_x) = {}^{2}_{m|}\bar{A}_x - ({}_{m|}\bar{A}_x)^2. \tag{34}
$$

Finally, we examine a varying life insurance payable at the moment of death. We have the following situation.

An insured is aged x at the time the insurance is taken out. The insurance pays $\$k$ at the moment of death, if death occurs in year k, and $k < n$. The number n is a fixed positive integer.

The present value of this insurance is denoted by $(I\bar{A})^{1}_{x:\overline{n|}}$. The present value of the cash flow is $g(T_x)$, where

$$g(t) = \begin{cases} k\, v^t & \text{if } t < n \\ 0 & \text{if } t \geq n. \end{cases} \tag{35}$$

We can express the present value of this insurance as

$$(I\bar{A})^{1}_{x:\overline{n|}} = \sum_{m=0}^{n-1} {}_{m|}\bar{A}^{1}_{x:\overline{n-m|}}$$

$$= \sum_{m=0}^{n-1} \frac{\bar{M}_{x+m} - \bar{M}_{x+n}}{D_x}.$$

Let us define the commutation function $\bar{R}_x$ by

$$\bar{R}_x = \sum_{i=0}^{\infty} \bar{M}_{x+k}. \tag{36}$$

Then, we get

$$(I\bar{A})^{1}_{x:\overline{n|}} = \frac{\bar{R}_x - \bar{R}_{x+n} - n\,\bar{M}_{x+n}}{D_x}, \tag{37}$$

$$(I\bar{A})_x = \frac{\bar{R}_x}{D_x}, \tag{38}$$

and

$$(I\bar{A})^{1}_{x:\overline{n|}} = (I\bar{A})_x - {}_nE_x\,(I\bar{A})_{x+n} - n\,{}_nE_x\,\bar{A}_{x+m}. \tag{39}$$

Using (24) and (25), we get the following approximations

$$\bar{R}_x \approx (1+i)^{\frac{1}{2}} R_x, \tag{40}$$

$$(I\bar{A})^{1}_{x:\overline{n}|} \approx (1+i)^{\frac{1}{2}} (IA)^{1}_{x:\overline{n}|} \tag{41}$$

and

$$(I\bar{A})_x \approx (1+i)^{\frac{1}{2}} (IA)_x. \tag{42}$$

EXAMPLE 3.4. A whole life insurance with a sum insured $3200 payable at the moment of death is taken out by a person aged 30. Obtain the present value of the insurance, using a 6% annual interest rate. Also, find the standard deviation of the insurance.

Solution: Using the approximation formulas, we get

$$\begin{aligned}\bar{A}_{30} \approx (1.06)^{\frac{1}{2}} A_{30} &= 1.0295630 \times 0.1024835 \\ &= 0.1055132.\end{aligned}$$

So, the present value is $3200 × 0.1055132 = $337.64. Furthermore,

$$\begin{aligned}V(\bar{A}_{30}) \approx 1.06\,({}^2A_{30} - A_{30}^2) &= 1.06(0.0253113 - (0.1024835)^2) \\ &= 0.0156970.\end{aligned}$$

Hence, the standard deviation is $3200 $\sqrt{0.0156970}$ = $400.92.

EXAMPLE 3.5. A person aged 40 takes out a life insurance that pays $1800 at the moment of death if death occurs after the age of 50. Find the present value of the insurance at an interest rate of 6% per annum. What is the standard deviation of the insurance?

Solution: The approximation formulas give

$$\begin{aligned}{}_{10|}\bar{A}_{40} \approx (1.06)^{\frac{1}{2}}\, {}_{10|}A_{40} = (1.06)^{\frac{1}{2}} \frac{M_{50}}{D_{40}} &\\ = 1.029563 \frac{1210.1957}{9054.46} &\\ = 1.029563 \times 0.1336574 &\\ = 0.1376087. &\end{aligned}$$

Therefore, the present value is $1800 × 0.1376087 = $247.70. We also have

$$V(_{10|}\bar{A}_{40}) \approx 1.06\, V(_{10|}A_{40}) = 1.06\,({}^2_{10|}A_{40} - (_{10|}A_{40})^2).$$

Now,

$$\begin{aligned}
{}^2_{10|}A_{40} &= v^{10}\,{}_{10}E_{40}\,{}^2A_{50} \\
&= v^{10}\frac{D_{50}}{D_{40}}\,{}^2A_{50} \\
&= 0.55839\,\frac{4859.30}{9054.46}\,0.0947561 \\
&= 0.0283959,
\end{aligned}$$

so

$$V(_{10|}\bar{A}_{40}) \approx 1.06(0.0283959 - (0.1336574)^2) = 0.0111635.$$

Thus, the standard deviation is $\$1800\,\sqrt{0.0111635} = \$190.18$.

EXAMPLE 3.6. A person aged 45 takes out a 15 year increasing temporary insurance. The death benefit payable at the moment of death is \$1000 in the first year and increases by \$200 each year. Find the present value of the insurance based on a 6% annual interest rate.

Solution: The present value of the insurance is

$$200(I\bar{A})^{1}_{45:\overline{15|}} + (1000 - 200)\,\bar{A}^{1}_{45:\overline{15|}}.$$

We have

$$\begin{aligned}
(I\bar{A})^{1}_{45:\overline{15|}} &\approx (1.06)^{\frac{1}{2}}\,(IA)^{1}_{45:\overline{15|}} \\
&= 1.06^{\frac{1}{2}}\,\frac{R_{45} - R_{60} - 15M_{60}}{D_{45}} \\
&= 1.0295630\,\frac{30723.7061 - 13459.2908 - 15 \times 916.2423}{6657.69} \\
&= 0.5444630
\end{aligned}$$

and

$$\bar{A}^{1}_{45:\overline{15|}} \approx (1.06)^{\frac{1}{2}}\,A^{1}_{45:\overline{15|}}$$

$$= (1.06)^{\frac{1}{2}} \frac{M_{45} - M_{60}}{D_{45}}$$
$$= 1.0295630 \frac{1339.5427 - 916.2423}{6657.69}$$
$$= 0.0654603.$$

Therefore, the present value is $200 \times 0.5444630 + 800 \times 0.0654603 = \$161.26$.

For the rest of the book, if we do not specify whether a death benefit is payable at the end of the year of death or at the moment of death, it will be understood that we mean the former.

PROBLEMS

3.1. Based on a 4% annual rate of interest, evaluate

a) C_{40}
b) C_{50}
c) $A^{\;1}_{30:\overline{2}|}$

3.2. Based on a 6% annual rate of interest, find
a) C_{35}
b) M_{45}
c) A_{50}
d) $V(A_{50})$
e) $A^{\;1}_{20:\overline{30}|}$
f) $V(A^{\;1}_{20:\overline{30}|})$
g) $_{10|}A_{35}$
h) $V(_{10|}A_{35})$
i) $_{5|}A^{\;1}_{40:\overline{20}|}$

3.3. A whole life insurance on a life aged 60 pays a death benefit of \$7000 at the end of the year of death. Based on a 6% annual rate of interest, determine the present value of the insurance. Also obtain the standard deviation of the insurance.

3.4. A deferred insurance issued to a life aged 45 pays \$8000 at the end of the year of death if death occurs after the age of 50. Based on a 6% annual rate of interest, find the present value and the standard deviation of the insurance.

3.5. A 10 year temporary insurance on a life aged 55 is purchased for \$320. Find the sum insured payable at the end of the year of death, if the annual interest rate is 6%.

3.6. Based on a 6% annual rate of interest, obtain

a) R_{40}
b) $(IA)_{50}$
c) $(IA)^{1}_{25:\overline{20|}}$

3.7. The death benefit of a 15 year temporary insurance on a life aged 35 is payable at the end of the year of death. The amount of the benefit is \$3000 in the first year and increases by \$200 each year. Find the present value of the insurance at a 6% annual rate of interest.

3.8. Based on a 6% annual rate of interest, obtain the approximate values of the following expressions.

a) $\bar{C}_{50}$

b) $\bar{M}_{35}$

c) $\bar{A}_{45}$

d) $V(\bar{A}_{45})$

e) $\bar{A}^{1}_{30:\overline{20|}}$

f) $V(\bar{A}^{1}_{30:\overline{20|}})$

g) $_{35|}\bar{A}_{20}$

h) $V(_{35|}\bar{A}_{20})$

i) $_{10|}\bar{A}^{1}_{50:\overline{15|}}$

3.9. The death benefit of a whole life insurance on a life aged 40 is payable at the moment of death. Determine the sum insured if the purchase price is \$560 and a 6% annual interest rate is used.

3.10. Find the present value of a 10 year temporary insurance issued to a life aged 40 with a death benefit of \$2000 payable at the moment

of death. Also, obtain the standard deviation of the insurance. Use a 6% annual interest rate.

3.11. An insurance on a life aged 50 pays \$7000 at the moment of death if death occurs after the age of 55. Based on a 6% annual rate of interest, find the present value and the standard deviation of the insurance.

3.12. A death benefit of an insurance is \$3000 payable at the moment of death if death occurs between the ages of 55 and 65. Find the present value of the insurance at the age of 40, based on a 6% annual rate of interest.

3.13. Based on a 6% annual rate of interest, find

a) $\bar{R}_{60}$

b) $(I\bar{A})_{40}$

c) $(I\bar{A})^{1}_{30:\overline{15}|}$

3.14. The death benefit of an insurance issued to a life aged 40 is payable at the moment of death. The death benefit is \$3500 in the first year and increases by \$100 each year. Find the present value of the insurance at a 6% annual interest rate.

3.4. ENDOWMENTS

An endowment insurance is the combination of a temporary insurance with a pure endowment payable at the end of the term of the insurance. We will focus on the following general situation.

The insurance is taken out at the age of x. A death benefit of \$1 is payable if the insured dies between the ages of $x+m$ and $x+m+n$, and a benefit of \$1 is paid at the age of $x+m+n$ on survival. The number m is a nonnegative integer and n is a positive integer.

Depending on the timing of the payment of the death benefit, we can distinguish between two types of endowments. One of them pays the death benefit at the end of the year of death and the other one pays at the moment of death. In any case, we can use (25) and (27) of Section 3.1 with I_1: n-year term insurance deferred for m years, I_2: pure endowment insurance payable at the age of $x+m+n$, and I_0: endowment insurance.

If the death benefit is payable at the end of the year of death, the

expected value of the endowment is denoted by ${}_{m|}A_{x:\overline{n}|}$. The present value of the cash flow can be expressed as $h(K_x)$, where

$$h(k) = \begin{cases} 0 & \text{if } k < m \\ v^{k+1} & \text{if } m \le k < m+n \\ v^{m+n} & \text{if } m+n \le k. \end{cases} \tag{1}$$

Using (27) of Section 3.1, we obtain

$${}_{m|}A_{x:\overline{n}|} = {}_{m|}A^{1}_{x:\overline{n}|} + A_{x:\overline{m+n}|}^{\;\;\;\;1}. \tag{2}$$

Thus, from (5) of Section 3.2 and (4) of Section 3.3, we get

$$\begin{aligned} {}_{m|}A_{x:\overline{n}|} &= \frac{M_{x+m} - M_{x+m+n}}{D_x} + \frac{D_{x+m+n}}{D_x} \\ &= \frac{M_{x+m} - M_{x+m+n} + D_{x+m+n}}{D_x}. \end{aligned} \tag{3}$$

Furthermore,

$$V({}_{m|}A_{x:\overline{n}|}) = {}^{2}_{m|}A_{x:\overline{n}|} - ({}_{m|}A_{x:\overline{n}|})^2. \tag{4}$$

If $m = 0$, we use the notation $A_{x:\overline{n}|}$. Then we have

$$A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;\;1}, \tag{5}$$

$$\begin{aligned} A_{x:\overline{n}|} &= \frac{M_x - M_{x+n}}{D_x} + \frac{D_{x+n}}{D_x} \\ &= \frac{M_x - M_{x+n} + D_{x+n}}{D_x}, \end{aligned} \tag{6}$$

and

$$V(A_{x:\overline{n}|}) = {}^{2}A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2. \tag{7}$$

If n is infinity (or more precisely, n is so large that $x + m + n > \omega$), the endowment insurance coincides with a whole life insurance since the insured

will definitely die before reaching the age of $x + m + n$. Thus, as n goes to infinity, we get

$$\lim_{n\to\infty} A_{x:\overline{n|}} = A_x$$

and

$$\lim_{n\to\infty} m\,|A_{x:\overline{n|}} = m\,|A_x.$$

If the death benefit is payable at the moment of death, the expected value of the endowment is denoted by $m|\,\overline{A}_{x:\overline{n|}}$. Then, the present value of the cash flow is $g(T_x)$, where

$$g(t) = \begin{cases} 0 & \text{if } t < m \\ v^t & \text{if } m \le t < m + n \\ v^{m+n} & \text{if } m + n \le t. \end{cases} \tag{8}$$

We have

$$m|\,\bar{A}_{x:\overline{n|}} = m|\,\bar{A}^{1}_{x:\overline{n|}} + A_{x:\overline{m+n|}}^{\quad\;\; 1} \tag{9}$$

so

$$m|\,\bar{A}_{x:\overline{n|}} = \frac{\bar{M}_{x+m} - \bar{M}_{x+m+n}}{D_x} + \frac{D_{x+m+n}}{D_x}$$

$$= \frac{\bar{M}_{x+m} - \bar{M}_{x+m+n} + D_{x+m+n}}{D_x}, \tag{10}$$

whose approximation can be obtained as

$$m|\,\bar{A}_{x:\overline{n|}} \approx (1 + i)^{\frac{1}{2}}\, m\,|A^{1}_{x:\overline{n|}} + A_{x:\overline{m+n|}}^{\quad\;\; 1}$$

$$= \frac{(1 + i)^{\frac{1}{2}} (M_{x+m} - M_{x+m+n}) + D_{x+m+n}}{D_x}. \tag{11}$$

It is important to remember that ${}_{m|}\bar{A}_{x:\overline{n}|}$ cannot be approximated by $(1+i)^{\frac{1}{2}}\, {}_{m|}A_{x:\overline{n}|}$.

For the variance, we get the following expression

$$V({}_{m|}\bar{A}_{x:\overline{n}|}) = {}^{2}_{m|}\bar{A}_{x:\overline{n}|} - ({}_{m|}\bar{A}_{x:\overline{n}|})^2. \tag{12}$$

If $m = 0$, we use the notation $\bar{A}_{x:\overline{n}|}$. Then, we have

$$\bar{A}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;\,1}, \tag{13}$$

$$\begin{aligned}\bar{A}_{x:\overline{n}|} &= \frac{\bar{M}_x - \bar{M}_{x+n}}{D_x} + \frac{D_{x+n}}{D_x}\\ &= \frac{\bar{M}_x - \bar{M}_{x+n} + D_{x+n}}{D_x}\end{aligned} \tag{14}$$

whose approximation is

$$\begin{aligned}\bar{A}_{x:\overline{n}|} &\approx (1+i)^{\frac{1}{2}} A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;\,1}\\ &= \frac{(1+i)^{\frac{1}{2}}(M_x - M_{x+n}) + D_{x+n}}{D_x}.\end{aligned} \tag{15}$$

For the variance we get

$$V(\bar{A}_{x:\overline{n}|}) = {}^{2}\bar{A}_{x:\overline{n}|} - (\bar{A}_{x:\overline{n}|})^2. \tag{16}$$

We also have

$$\lim_{n\to\infty} \bar{A}_{x:\overline{n}|} = \bar{A}_x$$

and

$$\lim_{n\to\infty} {}_{m|}\bar{A}_{x:\overline{n}|} = {}_{m|}\bar{A}_x.$$

EXAMPLE 4.1. A person aged 40 takes out an endowment insurance for 20 years with a benefit of \$5000. Find the present value and the standard deviation of the insurance

a) if the death benefit is paid at the end of the year of death,
b) if the death benefit is paid at the moment of death.

Use an annual interest rate of 6%.

Solution: a) First, we determine $A_{40:\overline{20}|}$ and $V(A_{40:\overline{20}|})$. We have

$$\begin{aligned} A_{40:\overline{20}|} &= \frac{M_{40} - M_{60} + D_{60}}{D_{40}} \\ &= \frac{1460.7038 - 916.2423 + 2482.16}{9054.46} \\ &= 0.3342686. \end{aligned}$$

Moreover,

$$V(A_{40:\overline{20}|}) = {}^2A_{40:\overline{20}|} - (A_{40:\overline{20}|})^2.$$

Now,

$${}^2A_{40:\overline{20}|} = {}^2A^{1}_{40:\overline{20}|} + {}^2A_{40:\overline{20}|}^{\;\;\;\;\;\;\;1}.$$

We can write

$$\begin{aligned} {}^2A^{1}_{40:\overline{20}|} &= {}^2A_{40} - v^{20}\,{}_{20}E_{40}\,{}^2A_{60} \\ &= {}^2A_{40} - v^{20}\frac{D_{60}}{D_{40}}\,{}^2A_{60} \\ &= 0.0486332 - 0.31180\,\frac{2482.16}{9054.46}\,0.1774113 \\ &= 0.0486332 - 0.0854758 \times 0.1774113 \\ &= 0.0334688, \end{aligned}$$

and

$${}^2A_{40:\overline{20}|}^{\;\;\;\;\;\;\;1} = v^{20}\,{}_{20}E_{40} = 0.0854758.$$

Therefore,

$$^2A_{40:\overline{20}|} = 0.0334688 + 0.0854758 = 0.1189446,$$

so

$$V(A_{40:\overline{20}|}) = 0.1189446 - (0.3342688)^2 = 0.0072090.$$

Thus, the present value of the insurance is $\$5000 \times 0.3342688 = \$1671.34$ and the standard deviation is

$$\$5000\sqrt{0.0072090} = \$424.53.$$

b) Using approximations, we get

$$\begin{aligned}\bar{A}_{40:\overline{20}|} &\approx \frac{(1.06)^{\frac{1}{2}}(M_{40} - M_{60}) + D_{60}}{D_{40}} \\ &= \frac{1.029563(1460.7038 - 916.2423) + 2482.16}{9054.46} \\ &= 0.3360463.\end{aligned}$$

Furthermore,

$$V(\bar{A}_{40:\overline{20}|}) = {}^2\bar{A}_{40:\overline{20}|} - (\bar{A}_{40:\overline{20}|})^2.$$

Now,

$$^2\bar{A}_{40:\overline{20}|} = {}^2\bar{A}^{\,1}_{40:\overline{20}|} + {}^2A_{40:\overline{20}|}^{\;\;\;\;\;1}.$$

We have

$$\begin{aligned}{}^2\bar{A}^{\,1}_{40:\overline{20}|} &= 1.06\ {}^2A^{1}_{40:\overline{20}|} \\ &= 1.06 \times 0.0334688 \\ &= 0.0354769\end{aligned}$$

so

$$ {}^2\bar{A}_{40:\overline{20}|} = 0.0354769 + 0.0854758 = 0.1209527. $$

Therefore,

$$ V(\bar{A}_{40:\overline{20}|}) = 0.1209527 - (0.3360463)^2 = 0.0080256. $$

So, the present value of the insurance is $\$5000 \times 0.3360463 = \$1680.23$ and the standard deviation is

$$ \$5000\ \sqrt{0.0080256} = \$447.93. $$

Note that, in line with what we said at the end of Section 3.3, the death benefit of an endowment is assumed to be payable at the end of the year of death, unless otherwise stated.

PROBLEMS

4.1. Based on a 6% annual rate of interest, obtain

a) $A_{30:\overline{10}|}$

b) $V(A_{30:\overline{10}|})$

c) $_{15|}A_{40:\overline{20}|}$

4.2. The death benefit of a 20 year endowment insurance of \$3000 is payable at the end of the year of death. Based on a 6% annual rate of interest, obtain the present value and the standard deviation of the insurance at the age of 30.

4.3. Based on a 6% annual rate of interest, obtain

a) $\bar{A}_{35:\overline{25}|}$

b) $V(\bar{A}_{35:\overline{25}|})$

c) $_{10|}\bar{A}_{25:\overline{30}|}$

4.4. The death benefit of a 15 year endowment insurance of \$4500 is payable at the moment of death. Based on a 6% annual rate of interest, find the present value and the standard deviation of the insurance at the age of 45.

3.5. Life Annuities

A life annuity provides a regular payment while the insured is alive. As in financial mathematics, we can talk about annuities-due; that is, annuities payable in advance, and annuities-immediate; that is, annuities payable in arrears. Annuities can be paid yearly, $p$thly, or continuously. There are deferred annuities and varying annuities as well. Annuities can be payable for the whole future lifetime of a person, but the payments of an annuity can also be limited to a certain number of years. If we do not specify the term of the annuity it is always understood that it is an annuity for life. If the term of the annuity is limited, it is called a temporary or term annuity.

Let us consider annuities-due in a general setting.

An insured is aged x at the time the insurance is taken out. The insurance pays \$1 at the beginning of each year between the ages of $x + m$ and $x + m + n$, if the insured is still alive. The number m is a fixed nonnegative integer and n is a fixed positive integer. This is called an n year temporary (or term) annuity-due deferred m years.

There are two ways of looking at this life annuity. The first approach is to regard it as the combination of pure endowment insurances. The first one pays \$1 at the age of $x + m$ if the insured is alive. The second one pays \$1 at the age of $x + m + 1$, if the insured survives to that age, etc. The last one pays \$1 at the age of $x + m + n - 1$ on survival.

So, denoting the present value of the cash flow by $g(T_x)$, using (1) of Section 3.2, we can express $g(t)$ as

$$g(t) = \sum_{k=m}^{m+n-1} g_k(t), \tag{1}$$

where

$$g_k(t) = \begin{cases} v^k & \text{if } t \geq k \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

We will denote the expected present value of this annuity-due by ${}_{m|}\ddot{a}_{x:\overline{n}|}$ or ${}_{m|n}\ddot{a}_x$. We will use the first notation in the book. Then, we have

$$ {}_{m|}\ddot{a}_{x:\overline{n}|} = \sum_{k=m}^{m+n-1} A_{x:\overline{k}|}^{\;\;1} \tag{3} $$

and from (5) of Section 3.2, we obtain

$$ {}_{m|}\ddot{a}_{x:\overline{n}|} = \sum_{k=m}^{m+n-1} \frac{D_{x+k}}{D_x}. \tag{4} $$

We introduce a new commutation function N_x, defined by

$$ N_x = \sum_{k=0}^{\infty} D_{x+k}. \tag{5} $$

The values of N_x at a 6% annual interest rate are tabulated in Appendix 2. Using this notation, we get

$$ {}_{m|}\ddot{a}_{x:\overline{n}|} = \frac{N_{x+m} - N_{x+m+n}}{D_x}. \tag{6} $$

Since the present value of no payments is zero, we define ${}_{m|}\ddot{a}_{x:\overline{0}|} = 0$. Note that this agrees with (6).

Although we have expressed the annuity as the sum of pure endowment insurances, we cannot use (29) of Section 3.1 to obtain the variance, since the function $g_i(t)\, g_j(t)$ $(i \neq j)$ is not identically zero.

However, using another approach, the variance can be obtained easily. As a byproduct, we will also find another formula for ${}_{m|}\ddot{a}_{x:\overline{n}|}$.

Note that (1) and (2) imply that $g(t)$ equals zero if $t < m$. If $m \le t < m + n$, we get

$$ g(t) = v^m + \ldots + v^{[t]} = v^m(1 + \ldots + v^{[t]-m}) = v^m \frac{1 - v^{[t]-m+1}}{1 - v} = \frac{v^m - v^{[t]+1}}{d}, $$

and if $t \ge m + n$, we obtain

$$ g(t) = v^m + \ldots + v^{m+n-1} = \frac{v^m - v^{m+n}}{d}. $$

In summary,

$$g(t) = \begin{cases} 0 & \text{if } t < m \\ \dfrac{v^m - v^{[t]+1}}{d} & \text{if } m \le t < m + n \\ \dfrac{v^m - v^{m+n}}{d} & \text{if } m + n \le t. \end{cases} \tag{7}$$

Moreover, since $[T_x] = K_x$, we have $g(T_x) = h(K_x)$, where

$$h(k) = \begin{cases} 0 & \text{if } k < m \\ \dfrac{v^m - v^{k+1}}{d} & \text{if } m \le k < m + n \\ \dfrac{v^m - v^{m+n}}{d} & \text{if } m + n \le k. \end{cases} \tag{8}$$

Note that $h(k)$ can be expressed as

$$h(k) = \frac{1}{d} \left(v^m - h_1(k) - h_2(k)\right), \tag{9}$$

where

$$h_1(k) = \begin{cases} 0 & \text{if } k < m \\ v^{k+1} & \text{if } m \le k < m + n \\ v^{m+n} & \text{if } m + n \le k, \end{cases} \tag{10}$$

and

$$h_2(k) = \begin{cases} v^m & \text{if } k < m \\ 0 & \text{if } m \le k. \end{cases} \tag{11}$$

Thus, $h_1(K_x)$ gives the present value of an n-year endowment insurance of \$1 deferred m years (see (1) of Section 3.4) and $h_2(K_x)$ is the present value of a life insurance with a death benefit of \$1 payable at the end of year m if the insured dies before that time. Let I_0 be the annuity defined by (8), I_1 the endowment defined by (10), and I_2 the life insurance defined by (11). Since I_1 is an endowment, we have

$$E(h_1(K_x)) = {}_{m|}A_{x:\overline{n}|}$$

and

$$V(h_1(K_x)) = {}^2_m|A_{x:\overline{n}|} - ({}_m|A_{x:\overline{n}|})^2.$$

On the other hand, we can find a relationship between I_2 and a pure endowment.

$$h_2(k) = v^m - h_3(k), \tag{12}$$

where

$$h_3(k) = \begin{cases} 0 & \text{if } k < m \\ v^m & \text{if } m \le k. \end{cases}$$

That means $h_3(K_x)$ gives the present value of a pure endowment of \$1 payable at the end of year n on survival. So, using (2) of Section 3.2, and (7) of Section 3.2, we get

$$E(h_2(K_x)) = v^m - E(h_2(K_x)) = v^m - A_{x:\overset{1}{\overline{m}|}}$$

and

$$\begin{aligned} V(h_2(K_x)) &= V(v^m - h_3(K_x)) = V(h_3(K_x)) \\ &= {}^2A_{x:\overset{1}{\overline{m}|}} - (A_{x:\overset{1}{\overline{m}|}})^2. \end{aligned}$$

Now, let us return to (9). Taking the expected values of both sides, we get

$$E(h(K_x)) = \frac{1}{d}\,(v^m - E(h_1(K_x)) - E(h_2(K_x))).$$

Thus

$${}_m|\ddot{a}_{x:\overline{n}|} = \frac{1}{d}\,(v^m - {}_m|A_{x:\overline{n}|} - (v^m - A_{x:\overset{1}{\overline{m}|}})).$$

Therefore,

$${}_m|\ddot{a}_{x:\overline{n}|} = \frac{1}{d}\,(A_{x:\overset{1}{\overline{m}|}} - {}_m|A_{x:\overline{n}|}). \tag{13}$$

If we want to find the variance of $h(K_x)$, we can ignore v^m since it is a constant term. So,

$$V(h(K_x)) = \frac{1}{d^2}\, V(h_1(K_x) + h_2(K_x)).$$

Furthermore, $h_1(k)\, h_2(k)$ is identically zero so we can use (29) of Section 3.1 to obtain the variance

$$V(m \mid \ddot{a}_{x:\overline{n}|}) = \frac{1}{d^2}\, ({}^2_m|A_{x:\overline{n}|} - (m|A_{x:\overline{n}|})^2 + {}^2A_{x:\overline{m}|}^{\;1} - (A_{x:\overline{m}|}^{\;1})^2$$
$$- 2(v^m - A_{x:\overline{m}|}^{\;1})\, m|A_{x:\overline{n}|}). \tag{14}$$

Formula (13) can be rewritten as

$$A_{x:\overline{m}|}^{\;1} = d\; m|\ddot{a}_{x:\overline{n}|} + m|A_{x:\overline{n}|}, \tag{15}$$

which can also be obtained by general reasoning. Assume a person aged x makes the following transaction with the insurance company. He/she pays the company \$1 at the age of $x + m$ if he/she survives to that age. Then the company will pay the yearly interest d of the \$1 in advance at the beginning of the following years. At the end of the year of death or at the age of $x + m + n$, whichever comes first, the company pays back the \$1 capital. Let us consider the present value of the cash flow at age x. Since the company pays the interest on \$1 every year before the capital is repaid, the present value of the sum of \$1 paid by the insured equals the sum of the present values of the interest payments and the capital repayment. Thus, taking the expected values of the present values at age x, we obtain (15).

If $m = 0$ and n is infinity (that is, $x + m + n > \omega$), we use the notation $\ddot{a}_x$. Since

$$v^0 = 1$$

and

$$A_{x:\overline{0}|}^{\;1} = v^0\, {}_0p_x = 1,$$

from (6), (13), and (14) we get

$$\ddot{a}_x = \frac{N_x}{D_x}, \tag{16}$$

$$\ddot{a}_x = \frac{1}{d}\,(1 - A_x), \tag{17}$$

and

$$V(\ddot{a}_x) = \frac{1}{d^2}\ ({}^2A_x - A_x^2). \tag{18}$$

The values of $\ddot{a}_x$ at a 6% annual interest rate are tabulated in Appendix 2.
Note that (17) implies

$$\frac{N_x}{D_x} = \frac{1}{d}\left(1 - \frac{M_x}{D_x}\right),$$

that is

$$D_x = dN_x + M_x. \tag{19}$$

Formula (19) is especially useful if we know the value of two of D_x, N_x, and M_x and we want to find the third one. The present values of other annuities can be expressed in terms of $\ddot{a}_x$. In view of (6) and (16), we have

$$\begin{aligned} {}_{m|}\ddot{a}_{x:\overline{n}|} &= \frac{D_{x+m}}{D_x}\frac{N_{x+m}}{D_{x+m}} - \frac{D_{x+m+n}}{D_x}\frac{N_{x+m+n}}{D_{x+m+n}} \\ &= {}_mE_x\, \ddot{a}_{x+m} - {}_{m+n}E_x\, \ddot{a}_{x+m+n}. \end{aligned} \tag{20}$$

If $m = 0$ and n is not infinity, we use the notation $\ddot{a}_{x:\overline{n}|}$. Then we get

$$\ddot{a}_{x:\overline{n}|} = \frac{N_x - N_{x+n}}{D_x}, \tag{21}$$

$$\ddot{a}_{x:\overline{n}|} = \frac{1}{d}\,(1 - A_{x:\overline{n}|}),$$

and

$$V(\ddot{a}_{x:\overline{n}|}) = \frac{1}{d^2}\ ({}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2). \tag{22}$$

If $m > 0$ and n is infinity, the notation is ${}_{m|}\ddot{a}_x$. We have

$${}_{m|}\ddot{a}_x = \frac{N_{x+m}}{D_x}, \tag{23}$$

$${}_{m|}\ddot{a}_x = \frac{1}{d}\,({}_mE_x - {}_{m|}A_x),$$

and

$$V(m\,|\,\ddot{a}_x) = \frac{1}{d^2}\left({}^2_m|A_x - (m\,|\,A_x)^2 + {}^2A^{\;1}_{x:\overline{m}|} - (A^{\;1}_{x:\overline{m}|})^2 - 2(v^m - A^{\;1}_{x:\overline{m}|})\, m\,|\,A_x\right). \tag{24}$$

Next, we want to examine varying annuities. Let us consider the following increasing annuity.

An insured is aged x at the time the insurance is taken out. The insurance pays $\$k$ at the beginning of year $k,$ if the insured is still alive and $k < n.$ The number n is a fixed positive integer.

The present value of this increasing annuity-due is denoted by $(I\ddot{a})_{x:\overline{n}|}$. This varying annuity can be expressed as the sum of level annuities.

$$(I\ddot{a})_{x:\overline{n}|} = \sum_{m=0}^{n-1} m\,|\,\ddot{a}_{x:\overline{n-m}|} = \sum_{m=0}^{n-1} \frac{N_{x+m} - N_{x+n}}{D_x}.$$

Defining the commutation function S_x by

$$S_x = \sum_{k=0}^{\infty} N_{x+k}, \tag{25}$$

we can write

$$(I\ddot{a})_{x:\overline{n}|} = \frac{S_x - S_{x+n} - n\,N_{x+n}}{D_x}. \tag{26}$$

The values of S_x at an annual interest rate of 6% are tabulated in Appendix 2. In n is infinity, from (26) we get

$$(I\ddot{a})_x = \frac{S_x}{D_x}. \tag{27}$$

Thus,

$$(I\ddot{a})_{x:\overline{n}|} = (I\ddot{a})_x - {}_nE_x\,(I\ddot{a})_{x+n} - n\,{}_nE_x\,\ddot{a}_{x+n}. \tag{28}$$

We have already seen six commutation functions: $D_x, C_x, N_x, M_x, R_x,$ and S_x. It is easier to remember their meaning if we note that C and $D,$ M and $N,$ and R and S are consecutive letters in the alphabet and the definitions also follow this order:

$$M_x = \sum_{k=0}^{\infty} C_{x+k}, \quad R_x = \sum_{k=0}^{\infty} M_{x+k},$$

and

$$N_x = \sum_{k=0}^{\infty} D_{x+k}, \quad S_x = \sum_{k=0}^{\infty} N_{x+k}.$$

EXAMPLE 5.1. What is the present value of a life annuity-due of $4000 per annum at the age of 50? Use an annual interest rate of 6%. Also find the standard deviation of the annuity.

Solution: First, we have to find $\ddot{a}_{50}$. From (16), we get

$$\ddot{a}_{50} = \frac{N_{50}}{D_{50}} = \frac{64467.45}{4859.30} = 13.26682.$$

We can also obtain $\ddot{a}_{50}$ from (17)

$$\ddot{a}_{50} = \frac{1}{d}(1 - A_{50}) = \frac{1}{0.056604}(1 - 0.2490475) = 13.26677.$$

If we look at the column headed $\ddot{a}_x$ in Appendix 2, we get $\ddot{a}_{50} = 13.26683$. So the present value of the insurance is $\$4000 \cdot 13.26683 = \$53067.32$.

Next, we compute $V(\ddot{a}_{50})$:

$$\begin{aligned} V(\ddot{a}_{50}) &= \frac{1}{d^2}\left({}^2A_{50} - A_{50}^2\right) \\ &= \frac{1}{(0.056604)^2}\left(0.0947561 - (0.2490475)^2\right) \\ &= 312.10861 \times 0.032731 \\ &= 10.21577. \end{aligned}$$

Thus, the standard deviation of the annuity is

$$\$4000\sqrt{10.21577} = \$12784.85\ .$$

EXAMPLE 5.2. A life annuity purchased at the age of 50 pays $1600 yearly in advance for a term of 20 years. Find the present value of the annuity at an annual interest rate of 6%. What is the standard deviation of the annuity?

Solution: First, we compute $\ddot{a}_{50:\overline{20}|}$:

$$\ddot{a}_{50:\overline{20}|} = \frac{N_{50} - N_{70}}{D_{50}} = \frac{64467.45 - 9597.05}{4859.30} = 11.29183.$$

So, the present value is $\$1600 \times 11.29183 = \$18066.93$. We also have

$$V(\ddot{a}_{50:\overline{20}|}) = \frac{1}{d^2}\,({}^2A_{50:\overline{20}|} - (A_{50:\overline{20}|})^2).$$

Now,

$$\begin{aligned} A_{50:\overline{20}|} &= \frac{M_{50} - M_{70} + D_{70}}{D_{50}} \\ &= \frac{1210.1957 - 576.7113 + 1119.94}{4859.30} \\ &= 0.3608389 \end{aligned}$$

which can also be obtained from

$$\begin{aligned} A_{50:\overline{20}|} &= 1 - d\ddot{a}_{50:\overline{20}|} \\ &= 1 - 0.056604 \times 11.29183 \\ &= 0.3608373. \end{aligned}$$

Now,

$${}^2A_{50:\overline{20}|} = {}^2A^{1}_{50:\overline{20}|} + {}^2A_{50:\overset{1}{\overline{20}|}}.$$

Since

$$\begin{aligned} {}^2A^{1}_{50:\overline{20}|} &= {}^2A_{50} - v^{20}\frac{D_{70}}{D_{50}}\,{}^2A_{70} \\ &= 0.0947561 - 0.31180 \times \frac{1119.94}{4859.30} \times 0.3064172 \\ &= 0.0727365 \end{aligned}$$

and

$${}^2A_{50:\overset{1}{\overline{20}|}} = v^{20}\frac{D_{70}}{D_{50}} = 0.31180\,\frac{1119.94}{4859.30} = 0.0718616,$$

we get

$${}^2A_{50:\overline{20}|} = 0.0727365 + 0.0718616 = 0.1445981.$$

Thus,

$$V(\ddot{a}_{50:\overline{20}|}) = 312.10861(0.1445981 - (0.3608373)^2) = 4.4926608.$$

Therefore, the standard deviation of the annuity is $\$1600\sqrt{4.4926608} =$ \$3391.34.

EXAMPLE 5.3. An annuity is payable yearly in advance while the insured is alive. The first payment is \$4000 and the payments increase by \$500 each year. Find the present value of this annuity if it is issued to a life aged 60. Use a 6% annual interest rate.

Solution: The present value of the annuity is

$$500(I\ddot{a})_{60} + (4000 - 500)\ddot{a}_{60}.$$

Now,

$$(I\ddot{a})_{60} = \frac{S_{60}}{D_{60}} = \frac{250959.22}{2482.16} = 101.10517$$

and

$$\ddot{a}_{60} = \frac{N_{60}}{D_{60}} = \frac{27664.55}{2482.16} = 11.14535.$$

So the present value of the annuity is $500 \times 101.10517 + 3500 \times 11.14535 =$ \$89561.31.

Next, we focus on annuities-immediate.

An insured is aged x at the time the insurance is taken out. The insurance pays \$1 at the end of each year between the ages of $x + m$ and $x + m + n$ if the insured is still alive. The number m is a fixed nonnegative integer and n is a fixed positive integer. This is called an n year temporary (or term) annuity-immediate deferred m years.

The notation for annuities-immediate can be obtained from those for annuities-due by dropping the two dots from above a.

Note that an n year annuity-immediate deferred m years is the same as an n year annuity-due deferred $m + 1$ years. Therefore,

$$_{m|}a_{x:\overline{n}|} = {}_{m|n}a_x = {}_{m+1|}\ddot{a}_{x:\overline{n}|} \qquad (29)$$

and using (6), we get

$$
{}_{m|}a_{x:\overline{n}|} = \frac{N_{x+m+1} - N_{x+m+n+1}}{D_x}. \tag{30}
$$

Using (8), we can express the present value of the cash flow by $h(K_x)$, where

$$
h(k) = \begin{cases} 0 & \text{if } k < m+1 \\ \dfrac{v^{m+1} - v^{k+1}}{d} & \text{if } m+1 \le k < m+n+1 \\ \dfrac{v^{m+1} - v^{m+n+1}}{d} & \text{if } m+n+1 \le k. \end{cases} \tag{31}
$$

From (14), we find the variance

$$
\begin{aligned}
V({}_{m|}a_{x:\overline{n}|}) &= V({}_{m+1|}\ddot{a}_{x:\overline{n}|}) \\
&= \frac{1}{d^2}\Big({}^{2}_{m+1|}A_{x:\overline{n}|} - ({}_{m+1|}A_{x:\overline{n}|})^2 + {}^{2}A_{x:\overline{m+1}|}^{\;\;\;1} - (A_{x:\overline{m+1}|}^{\;\;\;1})^2 \\
&\qquad - 2(v^{m+1} - A_{x:\overline{m+1}|}^{\;\;\;1})\, {}_{m+1|}A_{x:\overline{n}|}\Big).
\end{aligned} \tag{32}
$$

Note that a non-deferred $n+1$ year annuity-due of \$1 per annum is also equivalent to a non-deferred n year annuity-immediate of \$1 per annum combined with a payment of \$1 at the beginning of the first year. Denoting the present value of the annuity-immediate by $h(K_x)$ and the present value of the annuity-due by $h_1(K_x)$, we get

$$
h_1(K_x) = 1 + h(K_x). \tag{33}
$$

Taking expected value on both sides of (33), we get

$$
\ddot{a}_{x:\overline{n+1}|} = 1 + a_{x:\overline{n}|},
$$

thus

$$
a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n+1}|} - 1. \tag{34}
$$

Moreover, taking the variance on both sides of (33), we obtain

$$
V(\ddot{a}_{x:\overline{n+1}|}) = V(a_{x:\overline{n}|}), \tag{35}
$$

and using (22), we get

$$V(a_{x:\overline{n}|}) = \frac{1}{d^2}\left({}^2A_{x:\overline{n+1}|} - (A_{x:\overline{n+1}|})^2\right). \tag{36}$$

Let us derive formulas for some special choices of m and n. If $m = 0$ and n is infinity, we get

$$a_x = \frac{N_{x+1}}{D_x}, \tag{37}$$

$$a_x = \ddot{a}_x - 1,$$

and

$$V(a_x) = \frac{1}{d^2}\left({}^2A_x - A_x^2\right). \tag{38}$$

If $m = 0$ and n is not infinity, we obtain

$$a_{x:\overline{n}|} = \frac{N_{x+1} - N_{x+n+1}}{D_x}, \tag{39}$$

and

$$V(a_{x:\overline{n}|}) = \frac{1}{d^2}\left({}^2A_{x:\overline{n+1}|} - (A_{x:\overline{n+1}|})^2\right). \tag{40}$$

If $m > 0$ and n is infinity, we get

$$_{m|}a_x = \frac{N_{x+m+1}}{D_x}, \tag{41}$$

and

$$\begin{aligned} V(_{m|}a_x) = \frac{1}{d^2}\Big(& {}^2_{m+1|}A_x - (_{m+1|}A_x)^2 + {}^2A_{x:\overline{m+1}|}^{\;1} - (A_{x:\overline{m+1}|}^{\;1})^2 \\ & - 2(v^{m+1} - A_{x:\overline{m+1}|}^{\;1})\, _{m+1|}A_x\Big). \end{aligned} \tag{42}$$

We can also define increasing annuities-immediate.

An insured is aged x at the time the insurance is taken out. The insurance pays \$$k$ at the end of each year, if the insured is still alive and $k < n$. The number n is a fixed positive integer.

Note that this increasing annuity-immediate can be regarded as an increasing annuity-due deferred for one year. Thus using (20) of Section 3.1,

we get

$$(Ia)_{x:\overline{n}|} = {}_1E_x\,(I\ddot{a})_{x+1:\overline{n}|}$$
$$= \frac{D_{x+1}}{D_x}\,\frac{S_{x+1} - X_{x+n+1} - n\,N_{x+n+1}}{D_{x+1}}$$

so

$$(Ia)_{x:\overline{n}|} = \frac{S_{x+1} - S_{x+n+1} - n\,N_{x+n+1}}{D_x}. \tag{43}$$

If n is infinity, we get

$$(Ia)_x = \frac{S_{x+1}}{D_x}, \tag{44}$$

so

$$(Ia)_{x:\overline{n}|} = (Ia)_x - {}_nE_x\,(Ia)_{x+n} - n\,{}_nE_x\,a_{x+n}. \tag{45}$$

EXAMPLE 5.4. A life annuity-immediate of \$3000 per annum is purchased at the age of 60. Using a 6% annual interest rate find the present value of the annuity. What is the standard deviation of the annuity?

Solution: First, we have to determine the value of a_{60}. We can either write

$$a_{60} = \frac{N_{61}}{D_{60}} = \frac{25182.39}{2482.16} = 10.14535$$

or

$$a_{60} = \ddot{a}_{60} - 1 = 11.14535 - 1 = 10.14535.$$

So, the present value of the annuity is \$3000 × 10.14535 = \$30436.06. Next, we want to find the standard deviation. We have

$$V(a_{60}) = \frac{1}{d^2}\,({}^2A_{60} - A_{60}^2) = \frac{1}{(0.056604)^2}\,(0.1774113 - (0.369131)^2) = 12.84439.$$

Therefore, the standard deviation of the annuity is

$$\$3000\,\sqrt{12.84439} = \$10751.72.$$

EXAMPLE 5.5. A person aged 40 buys an annuity-immediate of \$2500 per annum, whose payments start at the end of year 10 and go on for 20 years, while the insured is alive. Find the present value of the annuity at a 6% annual rate of interest.

Solution: First, we compute $_{10|}a_{40}$:

$$_{10|}a_{40} = \frac{N_{51}}{D_{40}} = \frac{59608.16}{9054.46} = 6.58329.$$

Hence, the present value of the annuity is $\$2500 \times 6.58329 = \$16458.23$.

Next, we turn our attention to life annuities payable $p$thly. Let us focus on the following situation first.

An insured is aged x when the insurance is taken out. The insurance pays $\$\frac{1}{p}$ at the beginning of each $\frac{1}{p}$ long period, between the ages of $x+m$ and $x+m+n$, while the insured is alive. The number m is a nonnegative integer and n and p are positive integers.

The present value of this annuity-due payable $p$thly is denoted by $_{m|}\ddot{a}^{(p)}_{x:\overline{n}|}$ or $_{m|n}\ddot{a}^{(p)}_{x}$. We will use the first notation in the book. We can write

$$\begin{aligned} _{m|}\ddot{a}^{(p)}_{x:\overline{n}|} &= \frac{1}{p}\sum_{k=0}^{np-1} v^{m+\frac{k}{p}}\ {}_{m+\frac{k}{p}}p_x \\ &= \frac{1}{p}\sum_{k=0}^{np-1} \frac{v^{x+m+\frac{k}{p}}\ \ell_{x+m+\frac{k}{p}}}{v^x\ \ell_x} \\ &= \frac{1}{p}\sum_{k=0}^{np-1} \frac{D_{x+m+\frac{k}{p}}}{D_x}. \end{aligned} \tag{46}$$

The problem with this formula is that in most cases, we do not know the values of ℓ_x and D_x for noninteger x's. We could use an interpolation for ℓ_x but we will apply an approximation based on the Woolhouse formula. The approximation gives

$$\ddot{a}^{(p)}_{x} \approx \ddot{a}_x - \frac{p-1}{2p} - \frac{p^2-1}{12p^2}(\mu_x + \delta). \tag{47}$$

We will not prove the approximation here. The interested reader can find the proof in A. Neill: *Life Contingencies*, 1989.

The third term on the right-hand side of (47) is usually small compared

to the first two. Therefore, it is usually dropped, and the approximation

$$\ddot{a}_x^{(p)} \approx \ddot{a}_x - \frac{p-1}{2p} \tag{48}$$

is used. From (48), we can get an approximation for ${}_{m|}\ddot{a}_{\overline{n}|}^{(p)}$. We have

$$\begin{aligned}
{}_{m|}\ddot{a}_{x:\overline{n}|}^{(p)} &= {}_{m|}\ddot{a}_x^{(p)} - {}_{m+n|}\ddot{a}_x^{(p)} \\
&= {}_mE_x\, \ddot{a}_{x+m}^{(p)} - {}_{m+n}E_x\, \ddot{a}_{x+m+n}^{(p)} \\
&\approx {}_mE_x\left(\ddot{a}_{x+m} - \frac{p-1}{2p}\right) - {}_{m+n}E_x\left(\ddot{a}_{x+m+n} - \frac{p-1}{2p}\right) \\
&= {}_{m|}\ddot{a}_{x:\overline{n}|} - \frac{p-1}{2p}\,({}_mE_x - {}_{m+n}E_x).
\end{aligned} \tag{49}$$

So, if $m = 0$ and n is not infinity, we get

$$\ddot{a}_{x:\overline{n}|}^{(p)} \approx \ddot{a}_{x:\overline{n}|} - \frac{p-1}{2p}\,(1 - {}_nE_x). \tag{50}$$

Furthermore, if $m > 0$ and n is infinity, we obtain

$${}_{m|}\ddot{a}_x^{(p)} \approx {}_{m|}\ddot{a}_x - \frac{p-1}{2p}\,{}_mE_x\,. \tag{51}$$

We can also define increasing life annuities payable $p$thly. An insured is aged x when the insurance is taken out. The insurance pays $\$\frac{k}{p}$ at the beginning of each $\frac{1}{p}$ long period between the ages of $x + k - 1$ and $x + k$ on survival for every positive integer $k \le n$, where n is a fixed positive integer.

The present value of this increasing annuity-due payable $p$thly is denoted by $(I\ddot{a})_{x:\overline{n}|}^{(p)}$. Using (49), we get

$$\begin{aligned}
(I\ddot{a})_{x:\overline{n}|}^{(p)} &= \sum_{m=0}^{n-1} {}_{m|}\ddot{a}_{x:\overline{n-m}|}^{(p)} \\
&\approx \sum_{m=0}^{n-1} \left({}_{m|}\ddot{a}_x - \frac{p-1}{2p}\,({}_mE_x - {}_nE_x)\right)
\end{aligned}$$

$$= (I\ddot{a})_{x:\overline{n}|} - \frac{p-1}{2p}\left(\sum_{m=0}^{n-1} {}_mE_x - n\ {}_nE_x\right)$$

$$= (I\ddot{a})_{x:\overline{n}|} - \frac{p-1}{2p}(\ddot{a}_{x:\overline{n}|} - n\ {}_nE_x), \tag{52}$$

and if n is infinity, we have

$$(I\ddot{a})_x^{(p)} \approx (I\ddot{a})_x - \frac{p-1}{2p}\ddot{a}_x. \tag{53}$$

EXAMPLE 5.6. Find the present value at the age of 50 of a life annuity-due of \$300 per month. Use a 6% annual rate of interest.

Solution: The monthly installments of \$300 give a total payment of $\$300 \times 12 = \$3600$ per annum. Thus, the present value of the annuity is \$3600 $\ddot{a}_{50}^{(12)}$. We have

$$\begin{aligned}\ddot{a}_{50}^{(12)} &\approx \ddot{a}_{50} - \frac{12-1}{2 \cdot 12}\\ &= 13.26683 - 0.45833\\ &= 12.80850.\end{aligned}$$

Therefore, the present value is $\$3600 \times 12.8085 = \$46110.60$.

EXAMPLE 5.7. A life annuity pays \$3200 quarterly in advance until the age of 70. What is the present value of the annuity at the age of 55? Use a 6% annual rate of interest.

Solution: The total payment in one year is $\$3200 \times 4 = \$12800$.

Furthermore,

$$\begin{aligned}\ddot{a}_{55:\overline{70}|}^{(4)} &\approx \ddot{a}_{55:\overline{70}|} - \frac{4-1}{2 \times 4}(1 - {}_{15}E_{55})\\ &= \frac{N_{55} - N_{70}}{D_{55}} - \frac{3}{8}\left(1 - \frac{D_{70}}{D_{55}}\right)\\ &= \frac{43031.29 - 9597.05}{3505.37} - 0.375\left(1 - \frac{1119.94}{3505.37}\right)\\ &= 9.53801 - 0.25519 = 9.28282.\end{aligned}$$

Thus, the present value of the annuity is $\$12800 \times 9.28282 = \$118820.10$.

If the installments of the level annuity are paid in arrears, we get the following situation. An insured is aged x when the insurance is taken out.

The insurance pays $\$\frac{1}{p}$ at the end of each $\frac{1}{p}$ long period, between the ages of $x+m$ and $x+m+n$, while the insured is alive. The number m is a nonnegative integer and n and p are positive integers.

The present value of this annuity-immediate payable $p$thly is denoted by ${}_{m|}a^{(p)}_{x:\overline{n}|}$ or ${}_{m|n}a^{(p)}_x$. We will use the first notation in the book. We have

$$
\begin{aligned}
{}_{m|}a^{(p)}_{x:\overline{n}|} &= \frac{1}{p}\sum_{k=1}^{np} \frac{v^{x+m+\frac{k}{p}}\,\ell_{x+m+\frac{k}{p}}}{v^x\,\ell_x} \\
&= \frac{1}{p}\sum_{k=1}^{np} \frac{D_{x+m+\frac{k}{p}}}{D_x}. \qquad (54)
\end{aligned}
$$

Note that

$$\ddot{a}^{(p)}_x = \frac{1}{p} + a^{(p)}_x, \qquad (55)$$

thus from (47) and (55), we get the approximation

$$
\begin{aligned}
a^{(p)}_x &\approx \ddot{a}^{(p)}_x - \frac{1}{p} \approx \ddot{a}_x - \frac{p+1}{2p} - \frac{p^2-1}{12p^2}(\mu_x+\delta) \\
&= a_x + 1 - \frac{p+1}{2p} - \frac{p^2-1}{12p^2}(\mu_x+\delta) \\
&= a_x + \frac{p-1}{2p} - \frac{p^2-1}{12p^2}(\mu_x+\delta). \qquad (56)
\end{aligned}
$$

In practice, we use the approximation

$$a^{(p)}_x \approx a_x + \frac{p-1}{2p}. \qquad (57)$$

Following the steps of (49), from (57), we get the approximations

$${}_{m|}a^{(p)}_{x:\overline{n}|} \approx {}_{m|}a_{x:\overline{n}|} + \frac{p-1}{2p}\left({}_mE_x - {}_{m+n}E_x\right), \qquad (58)$$

$$a^{(p)}_{x:\overline{n}|} \approx a_{x:\overline{n}|} + \frac{p-1}{2p}\left(1 - {}_nE_x\right), \qquad (59)$$

and

$$_{m|}a_x^{(p)} \approx {}_{m|}a_x + \frac{p-1}{2p}\ {}_mE_x. \tag{60}$$

The increasing annuity-immediate payable $p$thly can be defined as follows. An insured is aged x when the insurance is taken out. The insurance pays $\$\frac{k}{p}$ at the end of each $\frac{1}{p}$ long period between the ages of $x + k - 1$ and $x + k$, on survival for every positive integer $k \le n$, where n is a fixed positive integer.

The present value of this increasing annuity-immediate payable $p$thly is denoted by $(Ia)_{x:\overline{n}|}^{(p)}$. By following the steps of (52), from (60) we get the approximation

$$(Ia)_{x:\overline{n}|}^{(p)} \approx (Ia)_{x:\overline{n}|} + \frac{p-1}{2p}\ (\ddot{a}_{x:\overline{n}|} - n\ {}_nE_x), \tag{61}$$

and

$$(Ia)_x^{(p)} \approx (Ia)_x + \frac{p-1}{2p}\ \ddot{a}_x. \tag{62}$$

EXAMPLE 5.8. A person aged 50 buys a life annuity that pays \$500 at the end of each month after the age of 60 is reached. Find the present value of the annuity at a 6% annual rate of interest.

Solution: This annuity pays $\$500 \times 12 = \$6000$ per annum. Moreover, we have

$$_{10|}a_{50}^{(12)} \approx {}_{10|}a_{50} + \frac{12-1}{2 \cdot 12}\ {}_{10}E_{50}.$$

Now,

$$_{10|}a_{50} = \frac{N_{61}}{D_{50}} = \frac{25182.39}{4859.30} = 5.18231$$

and

$$_{10}E_{50} = \frac{D_{60}}{D_{50}} = \frac{2482.16}{4859.30} = 0.51081,$$

thus,

$$_{10|}a_{50}^{(12)} = 5.18231 + \frac{11}{24}\ 0.51081 = 5.41643.$$

So, the present value of the annuity is $\$6000 \times 5.41643 = \$32498.58$.

EXAMPLE 5.9. A life annuity makes quarterly payments in arrears. In the first year, the quarterly payments are of amount $500 which increase by $100 each year. Find the present value of this varying annuity, if it is issued to a life aged 55. Use a 6% annual rate of interest.

Solution: The present value of the annuity is expressed by

$$4 \cdot 100(Ia)_{55}^{(4)} + 4\,(500 - 100)\, a_{55}^{(4)} = 400(Ia)_{55}^{(4)} + 1600\, a_{55}^{(4)}.$$

We have

$$\begin{aligned}(Ia)_{55}^{(4)} &\approx (Ia)_{55} + \frac{4-1}{2 \times 4}\, \ddot{a}_{55} \\ &= \frac{S_{56}}{D_{55}} + \frac{3}{8}\, \ddot{a}_{55} \\ &= \frac{390250.22}{3505.37} + \frac{3}{8}\, 12.27581 \\ &= 115.93268\end{aligned}$$

and

$$\begin{aligned}a_{55}^{(4)} &\approx a_{55} + \frac{4-1}{2 \times 4} = \ddot{a}_{55} - 1 + \frac{3}{8} \\ &= 12.27581 - 1 + 0.375 \\ &= 11.65081.\end{aligned}$$

So, the present value is $400 \times 115.93268 + 1600 \times 11.65081 = \$65014.37$.

Next, we study continuous life annuities.

An insured is aged x when the insurance is taken out. The insurance provides a continuous payment stream at a rate of $1 per annum, between the ages of $x + m$ and $x + m + n$, while the insured is alive. The number m is a nonnegative integer and n is a positive integer.

The present value of this continuous annuity is denoted by ${}_{m|}\bar{a}_{x:\overline{n}|}$ or ${}_{m|\overline{n}}\bar{a}_{x}$. We will use the first notation in the book. Using (11) of Section 3.1, we get

$${}_{m|}\bar{a}_{x:\overline{n}|} = \int_{m}^{m+n} v^t\, {}_tp_x\, dt$$

$$= \int_m^{m+n} \frac{D_{x+t}}{D_x}\, dt$$

$$= \frac{\int_m^{m+n} D_{x+t}\, dt}{D_x}$$

$$= \frac{\sum_{k=m}^{m+n-1} \int_k^{k+1} D_{x+t}\, dt}{D_x}. \tag{63}$$

Defining the commutation functions

$$\bar{D}_x = \int_0^1 D_{x+t}\, dt, \tag{64}$$

and

$$\bar{N}_x = \sum_{k=0}^{\infty} \bar{D}_{x+k} = \int_0^{\infty} D_{x+t}\, dt, \tag{65}$$

we obtain

$$_{m|}\bar{a}_{x:\overline{n|}} = \frac{\bar{N}_{x+m} - \bar{N}_{x+m+n}}{D_x}. \tag{66}$$

Some mortality tables contain the values of $\bar{D}_x$ and $\bar{N}_x$. However, if they are not available, we can approximate $\bar{D}_x$ by applying the trapezoid rule to the integral in (64):

$$\bar{D}_x \approx \frac{D_x + D_{x+1}}{2}. \tag{67}$$

Based on this, we get the following approximation for $\bar{N}_x$:

$$\bar{N}_x \approx \sum_{k=0}^{\infty} \frac{D_{x+k} + D_{x+k+1}}{2}$$

$$= -\frac{D_x}{2} + \sum_{k=0}^{\infty} D_{x+k}$$
$$= N_x - \frac{D_x}{2}. \tag{68}$$

Thus,

$$_{m|}\bar{a}_{x:\overline{n|}} \approx \frac{N_{x+m} - \frac{D_{x+m}}{2} - N_{x+m+n} - \frac{D_{x+m+n}}{2}}{D_x}$$
$$= {}_{m|}\ddot{a}_{x:\overline{n|}} - \frac{1}{2}({}_mE_x - {}_{m+n}E_x). \tag{69}$$

Note that the continuous annuity can be considered as the limit of an annuity (due or immediate) payable $p$thly as p goes to infinity.

Although we used different methods to obtain approximations for $_{m|}\ddot{a}^{(p)}_{x:\overline{n|}}$ and $_{m|}\bar{a}_{x+\overline{n|}}$, the limit of (49) gives (69) as p goes to infinity.

In order to find the variance of the continuous annuity, we have to express the present value of the cash flow in the form of $g(T_x)$. It follows from the definition of the continuous annuity that $g(t)$ equals zero if $t < m$. If $m \le t < m + n$, we get

$$g(t) = {}_{m|}\bar{a}_{\overline{t-m|}} = \frac{v^m - v^t}{\delta},$$

and if $t \ge m + n$, we obtain

$$g(t) = {}_{m|}\bar{a}_{\overline{n|}} = \frac{v^m - v^{m+n}}{\delta}.$$

In summary,

$$g(t) = \begin{cases} 0 & \text{if } t < m \\ \dfrac{v^m - v^t}{\delta} & \text{if } m \le t < m + n \\ \dfrac{v^m - v^{m+n}}{\delta} & \text{if } m + n \le t. \end{cases} \tag{70}$$

So, we can express $g(t)$ as

$$g(t) = \frac{1}{\delta}(v^m - g_1(t) - g_2(t)), \tag{71}$$

where

$$g_1(t) = \begin{cases} 0 & \text{if } t < m \\ v^t & \text{if } m \le t < m+n \\ v^{m+n} & \text{if } m+n \le t, \end{cases} \tag{72}$$

and

$$g_2(t) = \begin{cases} v^m & \text{if } t < m \\ 0 & \text{if } m \le t. \end{cases} \tag{73}$$

Note that $g_2(T_x) = h_2(K_x)$, where $h_2(K_x)$ is defined in (11). Furthermore, $g_1(T_x)$ is the present value of an n-year endowment insurance deferred m years (see (8) of Section 3.4). Note that the only difference between the endowment whose present value is $g_1(T_x)$ and the endowment whose present value is $h_1(K_x)$, defined in (10), is that while the death benefit of the former is payable at the moment of death, the death benefit of the latter is paid at the end of the year of death. So, following the steps that led to formulas (13) and (14), we obtain

$$m|\bar{a}_{x:\overline{n}|} = \frac{1}{\delta}\,(A^{\;1}_{x:\overline{m}|} - m|\bar{A}_{x:\overline{n}|}), \tag{74}$$

and

$$\begin{aligned} V(m|\bar{a}_{x:\overline{n}|}) = \frac{1}{\delta^2}\,\big({}^2_{m}|\bar{A}_{x:\overline{n}|} - (m|\bar{A}_{x:\overline{n}|})^2 + {}^2A^{\;1}_{x:\overline{m}|} \\ - (A^{\;1}_{x:\overline{m}|})^2 - 2(v^m - A^{\;1}_{x:\overline{m}|})\, m|\bar{A}_{x:\overline{n}|}\big). \end{aligned} \tag{75}$$

Rewriting (74), we obtain

$$A^{\;1}_{x:\overline{m}|} = \delta\, m|\bar{a}_{x:\overline{n}|} + m|\bar{A}_{x:\overline{n}|}. \tag{76}$$

It is left to the reader to prove (76) by general reasoning.

We can derive simpler formulas for special choices of m and n. If $m = 0$ and n is infinity, we obtain

$$\bar{a}_x = \frac{\bar{N}_x}{D_x},$$

$$\bar{a}_x \approx \ddot{a}_x - \frac{1}{2},$$

$$\bar{a}_x = \frac{1}{\delta}(1 - \bar{A}_x),$$

and

$$V(\bar{a}_x) = \frac{1}{\delta^2}({}^2\bar{A}_x - \bar{A}_x^2).$$

If $m = 0$ and n is not infinity, we get

$$\bar{a}_{x:\overline{n}|} = \frac{\bar{N}_x - \bar{N}_{x+n}}{D_x},$$

$$\bar{a}_{x:\overline{n}|} \approx \ddot{a}_{x:\overline{n}|} - \frac{1}{2}(1 - {}_nE_x),$$

$$\bar{a}_{x:\overline{n}|} = \frac{1}{\delta}(1 - \bar{A}_{x:\overline{n}|}),$$

and

$$V(\bar{a}_{x:\overline{n}|}) = \frac{1}{\delta^2}({}^2\bar{A}_{x:\overline{n}|} - (\bar{A}_{x:\overline{n}|})^2).$$

If $m > 0$ and n is infinity, we have

$$_{m|}\bar{a}_x = \frac{N_{x+m}}{D_x},$$

$$_{m|}\bar{a}_x \approx {}_{m|}\ddot{a}_x - \frac{1}{2}\,{}_mE_x,$$

$$_{m|}\bar{a}_x = \frac{1}{\delta}({}_mE_x - {}_{m|}\bar{A}_x),$$

and

$$V({}_{m|}\bar{a}_x) = \frac{1}{\delta^2}\left({}^2_{m|}\bar{A}_x - ({}_{m|}\bar{A}_x)^2 + {}^2A^{\;1}_{x:\overline{m}|} - (A^{\;1}_{x:\overline{m}|})^2 - 2(v^m - A^{\;1}_{x:\overline{m}|})\,{}_{m|}\bar{A}_x\right).$$

We can also examine varying continuous life annuities. Let us consider the following situation.

An insured is aged x when the insurance is taken out. The insurance provides a continuous payment stream at a rate of $\$k$ per annum in the year k for every positive integer $k \le n$, while the insured is still alive. The number n is a positive integer.

The present value of this increasing continuous life annuity is denoted by $(I\bar{a})_{x:\overline{n}|}$. Using (66), we get

$$(I\bar{a})_{x:\overline{n}|} = \sum_{m=0}^{n-1} {}_{m|}\bar{a}_{x:\overline{n-m}|} = \sum_{m=0}^{n-1} \frac{\bar{N}_{x+m} - \bar{N}_{x+n}}{D_x}.$$

Introducing the commutation function

$$\bar{S}_x = \sum_{k=0}^{\infty} \bar{N}_{x+k},$$

we obtain

$$(I\bar{a})_{x:\overline{n}|} = \frac{\bar{S}_x - \bar{S}_{x+n} - n\,\bar{N}_{x+n}}{D_x} \tag{77}$$

and

$$(I\bar{a})_x = \frac{\bar{S}_x}{D_x}. \tag{78}$$

Using the approximation (68), we get

$$\bar{S}_x \approx \sum_{k=0}^{\infty}\left(N_{x+k} - \frac{D_{x+k}}{2}\right) = S_x - \frac{N_x}{2}, \tag{79}$$

so

$$(I\bar{a})_{x:\overline{n}|} \approx \frac{S_x - \frac{N_x}{2} - \left(S_{x+n} - \frac{N_{x+n}}{2}\right) - n\left(N_{x+n} - \frac{D_{x+n}}{2}\right)}{D_x}$$
$$= (I\ddot{a})_{x:\overline{n}|} - \frac{1}{2}(\ddot{a}_{x:\overline{n}|} - n\ {}_nE_x), \tag{80}$$

and

$$(I\bar{a})_x \approx (I\ddot{a})_x - \frac{1}{2}\ddot{a}_x. \tag{81}$$

Note that (80) and (81) can also be obtained if we let p approach infinity in (52) and (53), respectively.

EXAMPLE 5.10. What is the present value at age 50 of a continuous life annuity payable at a rate of \$3600 per annum? Use a 6% annual interest rate. Also, find the standard deviation of the annuity.

Solution: First, we obtain $\bar{a}_{50}$:

$$\bar{a}_{50} \approx \ddot{a}_{50} - \frac{1}{2} = 13.26683 - 0.5 = 12.76683.$$

So, the present value is \$3600 × 12.76683 = \$45960.59. In order to find the standard deviation, we have to find $V(\bar{a}_{50})$ first.

$$V(\bar{a}_{50}) = \frac{1}{\delta^2}({}^2\bar{A}_{50} - \bar{A}_{50}^2)$$
$$= \frac{1}{\delta^2}1.06({}^2A_{50} - A_{50}^2)$$
$$= \frac{1}{(0.058269)^2}1.06(0.0947561 - (0.2490475)^2)$$
$$= 10.21871.$$

Thus, the standard deviation is $\$3600 \times \sqrt{10.21817} = \$11508.02$. Note that Example 5.6 described a similar situation with the difference that the payments were made monthly instead of continuously. Still, the present values in the two examples: \$45960.59 and \$46110.6 are close to each other. They differ from each other by less than 0.5%. That shows the continuous annuity gives a good approximation even in the case of monthly payments.

EXAMPLE 5.11. At life annuity pays \$6000 per annum continuously between the ages of 60 and 80. Determine the present value of the annuity

at the age of 50, using a 6% annual rate of interest.

Solution: We have to find ${}_{10|}\bar{a}_{50:\overline{20}|}$ first.

$${}_{10|}\bar{a}_{50:\overline{20}|} \approx {}_{10|}\ddot{a}_{50:\overline{20}|} - \frac{1}{2}({}_{10}E_{50} - {}_{30}E_{50}).$$

Now,

$$\begin{aligned}{}_{10|}\ddot{a}_{50:\overline{20}|} &= \frac{N_{60} - N_{80}}{D_{50}} \\ &= \frac{27664.55 - 2184.81}{4859.30} \\ &= 5.24350\end{aligned}$$

and

$$\begin{aligned}{}_{10}E_{50} - {}_{30}E_{50} &= \frac{D_{60} - D_{80}}{D_{50}} \\ &= \frac{2482.16 - 369.99}{4859.30} \\ &= 0.43467,\end{aligned}$$

so

$$\begin{aligned}{}_{10|}\bar{a}_{50:\overline{20}|} &\approx 5.24350 - \frac{1}{2}0.43467 \\ &= 5.02617.\end{aligned}$$

Therefore, the present value of the annuity is

$$\$6000 \times 5.02617 = \$30157.00.$$

Finally, in this chapter we introduce some special annuities.

The first annuity we are going to study makes guaranteed payments for a certain period of time, which thereafter become contingent on survival. Annuities of this type, called guaranteed annuities can be defined as follows.

An insured takes out an insurance for n years at the age of x. The insurance provides an annuity-certain of $1 per annum in the first m years, and for the rest of the term; that is, for $n - m$ years, a life annuity of $1 per annum is paid. The numbers n and m are positive integers ($m < n$).

Note that for any value of T_x, the present value of the cash flow (say $g(T_x)$) can be expressed as the present value of the m year annuity-certain (say c) plus the present value of an $n - m$ year life annuity deferred m years

(say $g_1(T_x)$). So

$$g(T_x) = c + g_1(T_x), \tag{82}$$

where c is a constant, not depending on T_x. Therefore, the present value of the insurance is

$$EPV = E(g(T_x)) = c + E(g_1(T_x)) \tag{83}$$

and the variance is

$$VPV = V(g(T_x)) = V(g_1(T_x)). \tag{84}$$

Depending on the type of the annuity, we get the folowing formulas.

In the case of an annuity-due, we have

$$EPV = \ddot{a}_{\overline{m}|} + {}_{m|}\ddot{a}_{x:\overline{n-m}|}$$

and

$$VPV = V({}_{m|}\ddot{a}_{x:\overline{n-m}|}),$$

so the variance can be calculated using (14).

If n is infinity, we use the notation $\ddot{a}_{\overline{\overline{x:m}}|}$ for the present value. Thus,

$$EPV = \ddot{a}_{\overline{\overline{x:m}}|} = \ddot{a}_{\overline{m}|} + {}_{m|}\ddot{a}_x$$

and

$$VPV = V(\ddot{a}_{\overline{\overline{x:m}}|}) = V({}_{m|}\ddot{a}_x).$$

In the case of an annuity-immediate, we get

$$EPV = a_{\overline{m}|} + {}_{m|}a_{x:\overline{n-m}|}$$

and

$$VPV = V({}_{m|}a_{x:\overline{n-m}|}).$$

Thus the variance can be obtained form (32).

Moreover, if n is infinity, then we have

$$EPV = a_{\overline{\overline{x:m}}|} = a_m + {}_{m|}a_x$$

and

$$VPV = V(a_{\overline{\overline{x:m}}|}) = V({}_{m|}a_x).$$

If the installments of the annuity-due are payable $p$thly, we get

$$EPV = \ddot{a}^{(p)}_{\overline{m}|} + {}_{m|}\ddot{a}^{(p)}_{x:\overline{n-m}|},$$

and if n is infinity, we have

$$EPV = \ddot{a}^{(p)}_{\overline{\overline{x:m}}|} = \ddot{a}^{(p)}_{\overline{m}|} + {}_{m|}\ddot{a}^{(p)}_x.$$

If the installments of the annuity-immediate are payable $p$thly, we obtain

$$EPV = a^{(p)}_{\overline{m}|} + {}_{m|}a^{(p)}_{x:\overline{n-m}|}$$

and if n is infinity, we get

$$EPV = a^{(p)}_{\overline{\overline{x:m}}|} = a^{(p)}_{\overline{m}|} + {}_{m|}a^{(p)}_x.$$

If the annuity is paid continuously, we have

$$EPV = \bar{a}_{\overline{m}|} + {}_{m|}\bar{a}_{x:\overline{n-m}|}$$

and

$$VPV = V({}_{m|}\bar{a}_{x:\overline{n-m}|}).$$

Hence, the variance can be computed using (75). Moreover, if n is infinity, we get

$$EPV = \bar{a}_{\overline{x:m}|} = \bar{a}_{\overline{m}|} + m|\bar{a}_x$$

and

$$VPV = V(\bar{a}_{\overline{x:m}|}) = V(m|\bar{a}_x).$$

EXAMPLE 5.12. A 20 year annuity-due of \$2500 per annum makes guaranteed payments in the first 5 years. Find the present value of the annuity at the age of 50 using a 6% annual rate of interest.

Solution: We need to find $\ddot{a}_{\overline{5}|} + 5|\ddot{a}_{50:\overline{15}|}$. Now,

$$\ddot{a}_{\overline{5}|} = 1 + a_{\overline{4}|} = 1 + 3.4651 = 4.4651$$

and

$$5|\ddot{a}_{50:\overline{15}|} = \frac{N_{55} - N_{70}}{D_{50}} = \frac{43031.29 - 9597.05}{4859.30} = 6.88046.$$

Thus, the present value is $2500(4.4651 + 6.88046) = \$28363.90$.

The next special annuity starts making payments after the insured dies. Since it is usually a member of the family who receives the payments of the annuity, the insurance is called a family income benefit (*FIB*).

In order to define a family income benefit, first we have to consider an n year annuity-certain of \$1 per annum deferred m years issued to a life aged x. The corresponding family income benefit consists of those payments of this annuity that are due after the death of the insured. The number m is a nonnegative integer and n, also referred to as the income term, is a positive integer.

Note that if we consider another insurance on a life aged x, an n year life annuity of \$1 per annum deferred m years, the two insurances "complement" each other; that is, the combined payments give an n year annuity-certain of \$1 per annum deferred m years. So for any value of T_x, the present value of the cash flow of the family income benefit (say $g(T_x)$) plus the present value of the n year life annuity deferred m years (say $g_1(T_x)$) equals the present value of an n year annuity-certain deferred m years (say c).

Thus,

$$g(T_x) = c - g_1(T_x), \tag{85}$$

where c is a constant that does not depend on T_x. So the present value of the *FIB* is

$$EPV = E(g(T_x)) = c - E(g_1(T_x)) \tag{86}$$

and the variance of the *FIB* is

$$VPV = V(g(T_x)) = V(g_1(T_x)). \tag{87}$$

Now, we give some formulas for family income benefits depending on the type of the payments of the annuity.

If we consider an annuity-immediate, we have

$$EPV = {}_{m|}a_{\overline{n}|} - {}_{m|}a_{x:\overline{n}|}$$

and

$$VPV = V({}_{m|}a_{x:\overline{n}|})$$

so the variance can be obtained from (32).

If we take $m = 0$, we get

$$EPV = a_{\overline{n}|} - a_{x:\overline{n}|}$$

and

$$VPV = V(a_{x:\overline{n}|}).$$

If the installments of the annuity-immediate are payable $p$thly, we obtain

$$EPV = {}_{m|}a^{(p)}_{\overline{n}|} - {}_{m|}a^{(p)}_{x:\overline{n}|}$$

and if $m = 0$, we have

$$EPV = a^{(p)}_{\overline{n}|} - a^{(p)}_{x:\overline{n}|}.$$

If the annuity is paid continuously, we get

$$EPV = {}_{m|}\bar{a}_{\overline{n}|} - {}_{m|}\bar{a}_{x:\overline{n}|}$$

and

$$VPV = V({}_{m|}\bar{a}_{x:\overline{n}|}),$$

so the variance can be calculated from (75).

Furthermore, if $m = 0$, we have

$$EPV = \bar{a}_{\overline{n}|} - \bar{a}_{x:\overline{n}|}$$

and

$$VPV = V(\bar{a}_{x:\overline{n}|}).$$

Note that we usually do not base a family income benefit on an annuity-due since this would not provide a payment in the time interval in which the insured dies. So if we do not say it otherwise, by a family income benefit we will mean a *FIB* based on an annuity-immediate.

EXAMPLE 5.13. A person aged 40 buys a family income benefit for an income term of 20 years. Find the present value of the insurance at a 6% annual rate of interest if

a) the benefit is \$500 per month.

b) the benefit is paid continuously at a rate of \$6000 per annum.

Solution: a) This insurance provides an annual benefit of $\$500 \times 12 = \$6000$. The present value is $\$6000\,(a^{(12)}_{\overline{20}|} - a^{(12)}_{40:\overline{20}|})$. We have

$$a^{(12)}_{\overline{20}|} = \frac{i}{i^{(12)}}\, a_{\overline{20}|} = 1.027211 \times 11.4699 = 11.78201$$

and

$$a^{(12)}_{40:\overline{20}|} \approx a_{40:\overline{20}|} + \frac{12 - 1}{2 \times 12}(1 - {}_{20}E_{40})$$

$$= \frac{N_{41} - N_{61}}{D_{40}} + \frac{11}{24}\left(1 - \frac{D_{60}}{D_{40}}\right)$$

$$= \frac{125101.93 - 25182.39}{9054.46} + \frac{11}{24}\left(1 - \frac{2482.16}{9054.46}\right)$$

$$= 11.03539 + 0.33269$$
$$= 11.36808.$$

Thus, the present value is $\$6000(11.78201 - 11.36808) = \$2483.58$.

b) In the continuous case, the present value can be obtained as

$\$6000(\bar{a}_{\overline{20}|} - \bar{a}_{40:\overline{20}|})$. Here, we get

$$\bar{a}_{\overline{20}|} = \frac{i}{\delta} a_{\overline{20}|} = 1.029709 \times 11.4699 = 11.81066$$

and

$$\bar{a}_{40:\overline{20}|} \approx \ddot{a}_{40:\overline{20}|} - \frac{1}{2}(1 - {}_{20}E_{40})$$

$$= \frac{N_{40} - N_{60}}{D_{40}} - \frac{1}{2}\left(1 - \frac{D_{60}}{D_{40}}\right)$$

$$= \frac{134156.39 - 27664.55}{9054.46} - \frac{1}{2}\left(1 - \frac{2482.16}{9054.46}\right)$$

$$= 11.76126 - 0.36293$$
$$= 11.39833.$$

Hence, the present value is $\$6000(11.81066 - 11.39833) = \$2473.98$. We can see that the difference between the present value of the benefit payable monthly and that of the benefit payable continuously less than 0.5%, which is relatively small. So, the computations based on a continuous payment come very close to the results of the monthly payments.

In order to introduce the third special annuity, let us consider an annuity-immediate. If the annuity is payable yearly and the insured dies a couple of days before the end of the year, the payment for that year will not be paid. It could be argued that this is not fair since the year's payment would have been made, had the insured survived a little longer. So, it seems reasonable to provide an additional payment on death proportional to the time elapsed since the last annuity payment.

Therefore, we define the following insurance, called a complete annuity-due.

An insured is aged x at the time the insurance is taken out. The insurance pays $\$\frac{1}{p}$ at the end of each $\frac{1}{p}$ long period, between the ages of $x+m$ and $x+m+n$, while the insured is still alive. Moreover, if death occurs between the ages of $x+m$ and $x+m+n$, and r is the time elapsed since the last annuity payment, an amount of $\$r$ is paid at the moment of death.

The present value of the complete annuity-due is denoted by ${}_{m|}\mathring{a}^{(p)}_{x:\overline{n}|}$ or ${}_{m\,|\,n}\mathring{a}^{(p)}_{x}$. We will use the first notation in the book. Let us divide the present value of the complete annuity-due into two parts. One is the present value of an annuity-due payable $p$thly; that is, ${}_{m|}a^{(p)}_{x:\overline{n}|}$. The other part is the present value of the benefit payable on death. The latter can be expressed as follows:

$$EPV_1 = \int_m^{m+n} r(t)\, v^t f_x(t)dt$$

$$= \int_m^{m+n} r(t)\, v^t\, {}_tp_x\, \mu_{x+t}\, dt, \tag{88}$$

where

$$r(t) = t - \frac{1}{p}[t \cdot p].$$

So, $r(t)$ is a function whose range is the interval between 0 and $\frac{1}{p}$. We will replace $r(t)$ by the midpoint of this interval, $\frac{1}{2p}$ in (88) to get the following approximation

$$EPV_1 \approx \frac{1}{2p} \int_m^{m+n} v^t\, {}_tp_x\, \mu_{x+t}\, dt$$

$$= \frac{1}{2p}\; {}_{m|}\overset{-1}{A}_{x:\overline{n}|}.$$

Hence, we obtain

$${}_{m|}\mathring{a}^{(p)}_{x:\overline{n}|} \approx {}_{m|}a^{(p)}_{x:\overline{n}|} + \frac{1}{2p}\; {}_{m|}\overset{-1}{A}_{x:\overline{n}|}. \tag{89}$$

With special choices of m and n, we get

$$\mathring{a}_x^{(p)} \approx a_x^{(p)} + \frac{1}{2p} \bar{A}_x, \tag{90}$$

$$\mathring{a}_{x:\overline{n}|}^{(p)} \approx a_{x:\overline{n}|}^{(p)} + \frac{1}{2p} \bar{A}^{1}_{x:\overline{n}|}, \tag{91}$$

and

$$m|\mathring{a}_x^{(p)} \approx m|a_x^{(p)} + \frac{1}{2p}\, m|\bar{A}_x \,. \tag{92}$$

If $p = 1$, we get

$$\mathring{a}_x \approx a_x + \frac{1}{2} \bar{A}_x, \tag{93}$$

$$\mathring{a}_{x:\overline{n}|} \approx a_{x:\overline{n}|} + \frac{1}{2} \bar{A}^{1}_{x:\overline{n}|}, \tag{94}$$

and

$$m|\mathring{a}_x \approx m|a_x + \frac{1}{2}\, m|\bar{A}_x. \tag{95}$$

EXAMPLE 5.14. A complete annuity-due of \$3000 per quarter is issued to a life aged 60. Find the present value of the annuity based on a 6% annual interest rate.

Solution: The annual payment of the annuity is $\$3000 \times 4 = \$12000$. So the present value can be expressed as

$$\$12000\left(a_{60}^{(4)} + \frac{1}{2 \times 4} \bar{A}_{60}\right).$$

Now

$$\begin{aligned} a_{60}^{(4)} &\approx a_{60} + \frac{4 - 1}{2 \times 4} \\ &= \ddot{a}_{60} - 1 + \frac{3}{8} \\ &= 11.14535 - 1 + 0.375 \end{aligned}$$

$$= 10.52035$$

and

$$\bar{A}_{60} \approx 1.06^{\frac{1}{2}} A_{60}$$
$$= 1.029563 \times 0.3691310$$
$$= 0.38004.$$

Therefore, the present value of the complete annuity is

$$\$12000\left(10.52035 + \frac{1}{8}0.38004\right) = \$126814.26.$$

PROBLEMS

5.1. Prove the following identities:

a) $m|\ddot{a}_{x:\overline{n}|} = \ddot{a}_{x:\overline{m+n}|} - \ddot{a}_{x:\overline{m}|}$

b) $m|\ddot{a}_{x:\overline{n}|} = {}_mE_x \cdot \ddot{a}_{x+m:\overline{n}|}$

5.2. Based on a 6% annual interest rate, evaluate

a) N_{35}
b) $\ddot{a}_{25}$
c) $V(\ddot{a}_{25})$
d) $\ddot{a}_{45:\overline{15}|}$
e) $V(\ddot{a}_{45:\overline{15}|})$
f) $25|\ddot{a}_{30}$
g) $5|\ddot{a}_{25:\overline{10}|}$

5.3. A life annuity-due of \$2000 per annum is issued to a life aged 30. Find the present value of the annuity, based on a 6% annual rate of interest. Also find the standard deviation of the annuity.

5.4. A temporary life annuity of \$2500 per annum is payable yearly in advance between the ages of 35 and 50. Find the present value of the annuity at the age 35, based on a 6% annual rate of interest. Also find the standard deviation of the annuity.

5.5. A life annuity-due of \$1500 per annum is payable after the age of 60. Find the present value of the annuity at the age of 40, based on a 6% annual rate of interest.

5.6. A temporary life annuity of \$3000 per annum payable yearly in advance, between the ages of 55 and 70, is purchased for a life aged 50. Find the present value of the annuity, based on a 6% annual rate of interest.

5.7. Based on a 6% annual rate of interest, find

a) S_{40}
b) $(I\ddot{a})_{35}$
c) $(I\ddot{a})_{40:\overline{15}|}$

5.8. A 20 year life annuity-due issued to a life aged 50 pays \$1500 in the first year and increases by \$100 each year. Find the present value of the annuity based on a 6% annual rate of interest.

5.9. Prove the following identities:

a) $m|a_{x:\overline{n}|} = a_{x:\overline{m+n}|} - a_{x:\overline{m}|}$,

b) $m|a_{x:\overline{n}|} = {}_mE_x \cdot a_{x+m} - {}_{m+n}E_x \cdot a_{x+m+n}$,

c) $m|a_{x:\overline{n}|} = {}_mE_x \cdot a_{x+m:\overline{n}|}$,

d) $m|a_{x:\overline{n}|} = m|\ddot{a}_{x:n+1} - {}_mE_x$.

5.10. If we look at (17), we might think a similar identity holds for annuities-immediate. Show that this is not true; that is,
$a_x \neq \frac{1}{i}\,(1 - A_x)$.

5.11. Based on a 6% annual rate of interest, find

a) a_{35}
b) $V(a_{35})$
c) $a_{25:\overline{10}|}$
d) $V(a_{25:\overline{10}|})$
e) $20|a_{40}$

f) $_{10|}a_{45:\overline{20}|}$

5.12. A life annuity-immediate of \$1500 per annum is issued to a life aged 45. Find the present value and the standard deviation of the annuity based on a 6% annual interest rate.

5.13. A temporary life annuity of \$2000 per annum is payable yearly in arrears between the ages of 40 and 55. Find the present value of the annuity at the age of 35 at a 6% annual rate of interest.

5.14. Based on a 6% annual rate of interest, obtain

a) $(Ia)_{20}$,

b) $(Ia)_{60:\overline{15}|}$.

5.15. A 25 year life annuity purchased for a life aged 40 makes a payment of \$2000 at the end of the first year and increases by \$150 each year. Find the present value of the annuity, based on a 6% annual interest rate.

5.16. Prove the following identities:

a) $_{m|}\ddot{a}^{(p)}_{x:\overline{n}|} = \ddot{a}^{(p)}_{x:\overline{m+n}|} - \ddot{a}^{(p)}_{x:\overline{m}|}$,

b) $_{m|}\ddot{a}^{(p)}_{x:\overline{n}|} = {}_mE_x \cdot \ddot{a}^{(p)}_{x+m} - {}_{m+n}E_x \cdot \ddot{a}^{(p)}_{x+m+n}$,

c) $_{m|}\ddot{a}^{(p)}_{x:\overline{n}|} = {}_mE_x \cdot \ddot{a}^{(p)}_{x+m:\overline{n}|}$.

5.17. Based on a 6% annual interest rate, obtain

a) $\ddot{a}^{(12)}_{30}$

b) $\ddot{a}^{(4)}_{40:\overline{20}|}$

c) $_{10|}\ddot{a}^{(2)}_{\overline{25}|}$

d) $_{10|}\ddot{a}^{(4)}_{35:\overline{15}|}$.

5.18. Find the present value of a life annuity of \$4000 per annum payable quarterly in advance issued to a life aged 35. Use a 6% annual rate of interest.

5.19. A 10 year life annuity pays \$800 monthly in advance. Determine the present value of the annuity at the age of 55 based on a 6%

annual interest rate.

5.20. Based on a 6% annual rate of interest, determine

a) $(I\ddot{a})^{(2)}_{45}$

b) $(I\ddot{a})^{(12)}_{55:\overline{10}|}$.

5.21. A life annuity payable monthly in advance is purchased for a life aged 20. The annual payment is \$3000 in the first year and increases by \$600 each year. Find the present value of the annuity, based on a 6% annual rate of interest.

5.22. Prove the following identities:

a) $m|a^{(p)}_{x:\overline{n}|} = a^{(p)}_{x:\overline{m+n}|} - a^{(p)}_{x:\overline{m}|}$

b) $m|a^{(p)}_{x:\overline{n}|} = {}_mE_x \cdot a^{(p)}_{x+m} - {}_{m+n}E_x \cdot a^{(p)}_{x+m+n}$,

c) $m|a^{(p)}_{x:\overline{n}|} = {}_mE_x \cdot a^{(p)}_{x+m:\overline{n}|}$,

d) $m|a^{(p)}_{x:\overline{n}|} = m|\ddot{a}^{(p)}_{x:\overline{n}|} - {}_mE_x + {}_{m+n}E_x$.

5.23. Based on a 6% annual rate of interest, evaluate

a) $a^{(2)}_{50}$

b) $a^{(4)}_{30:\overline{25}|}$

c) $10|a^{(12)}_{55}$

d) $25|a^{(12)}_{40:\overline{10}|}$

5.24. Find the present value of a life annuity paying \$300 monthly in arrears purchased for a life aged 40. Use a 6% annual rate of interest.

5.25. A life annuity of \$12000 per annum payable quarterly in arrears between the ages of 50 and 65, is issued to a life aged 45. Obtain the present value of the annuity at a 6% annual rate of interest.

5.26. Based on a 6% annual rate of interest, obtain

a) $(Ia)^{(4)}_{30}$

b) $(Ia)^{(2)}_{40:\overline{10}|}$

5.27. A 20 year life annuity makes quarterly payments in arrears. In the first year, the quarterly payment is \$600 and it increases by \$50 each year. Find the present value of the annuity at the age of 40, using a 6% annual interest rate.

5.28. Prove the following identities:

a) $_{m|}\bar{a}_{x:\overline{n}|} = \bar{a}_{x:\overline{m+n}|} - \bar{a}_{x:\overline{m}|}$,

b) $_{m|}\bar{a}_{x:\overline{n}|} = {}_mE_x \cdot \bar{a}_{x+m} - {}_{m+n}E_x \cdot \bar{a}_{x+m+n}$,

c) $_{m|}\bar{a}_{x:\overline{n}|} = {}_mE_x \cdot \bar{a}_{x+m:\overline{n}|}$.

5.29. Prove (76) by general reasoning.

5.30. Based on a 6% annual rate of interest, evaluate

a) $\bar{N}_{40}$

b) $\bar{a}_{20}$

c) $V(\bar{a}_{20})$

d) $\bar{a}_{30:\overline{25}|}$

e) $V(\bar{a}_{30:\overline{25}|})$

f) $_{10|}\bar{a}_{40}$

g) $_{15|}\bar{a}_{25:\overline{30}|}$

5.31. A continuous life annuity of \$3600 per annum is issued to a life aged 40. Find the present value of the annuity based on a 6% annual rate of interest. Also find the standard deviation of the annuity.

5.32. A 30 year life annuity payable continuously at a rate of \$4000 per annum is purchased for a life aged 20. Determine the present value of the annuity at a 6% annual rate of interest.

5.33. A continuous life annuity of \$2000 per annum issued to a life aged 50

is deferred 10 years. Determine the present value of the annuity, based on a 6% annual interest rate.

5.34. Based on a 6% annual rate of interest, obtain

a) $\bar{s}_{65}$

b) $(I\bar{a})_{30}$

c) $(I\bar{a})_{55:\overline{10}|}$.

5.35. A varying continuous life annuity is issued to a life aged 40. The rate of payment is \$3500 per annum in the first year and increases by \$400 each year. Find the present value of the annuity at a 6% annual rate of interest.

5.36. The rate of payment for a 15 year continuous life annuity is \$1200 per annum in the first year and increases by \$200 each year. Determine the present value of the annuity at the age of 35, based on a 6% annual interest rate.

5.37. A 25 year annuity-immediate of \$1000 per annum with guaranteed payments in the first 10 years is issued to a life aged 45. Find the present value of the annuity based on a 6% annual interest rate.

5.38. The payments of a 20 year annuity-due of \$800 per month are guaranteed in the first 10 years. Find the present value of the annuity at the age of 40, based on a 6% annual rate of interest.

5.39. A 25 year family income benefit, payable monthly, is purchased by a person aged 30 for \$3000. Determine the amount of the monthly payments based on a 6% annual rate of interest.

5.40. A 15 year family income benefit is payable continuously at a rate of \$8000 per annum. Find the present value of the insurance at the age of 45 based on a 6% annual interest rate.

5.41. A 10 year complete annuity-due of \$1500 per month is issued to a life aged 50. Determine the present value of the annuity using a 6% annual rate of interest.

5.42. Find the present value of a 15 year complete annuity-due of \$400 per quarter purchased for a life aged 30, based on a 6% annual rate of interest.

CHAPTER 4

PREMIUMS

The premium charged by an insurance company for an insurance policy serves two purposes. Firstly, it has to provide the funds needed to meet the benefit paying liabilities and secondly it has to cover the expenses of running the business. If we ignore the expenses and derive a premium based on the benefits only, we call it the net or risk premium. If the expenses are also included in the computation, we get the gross or office premium. It is important to point out that the net premium is uniquely determined by the mortality and the interest, factors that are beyond the control of an insurance company. On the other hand, expenses can be changed more easily: staff can be laid off, office space reduced, expenses reallocated to other insurance products, etc.

We will study two ways of paying the premium. The first possibility is to pay a single premium at the time the insurance is taken out. The second method involves the policyholder paying premiums regularly during a given time period. In the second case, it has to be taken into account that the premium payments will stop if the policy holder dies.

4.1. Net Premiums

In order to find the net (or risk) premium, we use the following equation:

$$\begin{aligned}&\text{Expected present value of the net premiums}\\&\quad = \text{Expected present value of the benefits.}\end{aligned} \tag{1}$$

The present values are usually determined at the time the insurance is issued. If we have a single premium, denoted by P, from (1), we obtain

$$P = \text{Expected present value of the benefits.} \tag{2}$$

That means, all the present values for the respective insurances discussed in Chapter 3 can also be interpreted as the net single premiums of the insurances. These insurances can also be combined, as the following example shows.

EXAMPLE 1.1. The following insurance is issued to a life aged 40. If the insured dies within 25 years, a death benefit of $\$6000$ is payable at the end of the year of death. Otherwise, a sum of $\$2000$ is payable yearly in advance from the age of 65.

Find the net single premium for the insurance at a 6% annual rate of interest.

Find the net single premium for the insurance at a 6% annual rate of interest.

Solution: The premium equals the present value of the insurance, so

$$P = 6000\, A^{\;1}_{40:\overline{25}|} + 2000\; {}_{25|}\ddot{a}_{40}.$$

Now,

$$A^{\;1}_{40:\overline{25}|} = \frac{M_{40} - M_{65}}{D_{40}} = \frac{1460.7038 - 750.5749}{9054.46} = 0.0784286$$

and

$${}_{25|}\ddot{a}_{40} = \frac{N_{65}}{D_{40}} = \frac{16890.50}{9054.46} = 1.865434.$$

Therefore, the premium is

$$P = 6000 \times 0.0784286 + 2000 \times 1.865434 = \$4201.44.$$

Next, we examine the other method of paying the premiums; that is, when the premium is paid in installments during a given time period. If the insured dies, the premium payments are terminated.

We want to focus on level premiums assuming the installments form an annuity. If P is the annual premium, we can express it in terms of the present value of the benefits and the present value of an annuity of $\$1$ per annum corresponding to the premium payments as

$$P \times (\text{Expected present value of a premium annuity of } \$1 \text{ per annum}) \\ = \text{Expected present value of the benefits,}$$

and therefore

$$P = \frac{\text{Expected present value of the benefits}}{\text{Expected present value of a premium annuity of } \$1 \text{ per annum}}.$$

If we do not specify the premium annuity more closely, it will be understood that it is an annuity-due. If the premium paying period coincides with the term of the insurance, we use some special notations. We will study this situation first.

In the case of a pure endowment of $\$1$, we denote the annual premium by $P_{x:\overline{n}|}^{\;\;\;1}$ or $P(A_{x:\overline{n}|}^{\;\;\;1})$ and hence,

$$P_{x:\overline{n}|}^{\;1} = P(A_{x:\overline{n}|}^{\;1}) = \frac{A_{x:\overline{n}|}^{\;1}}{\ddot{a}_{x:\overline{n}|}}.$$

The annual premium for an n year term insurance with a death benefit of $1 payable at the end of the year of death, is denoted by $P^1_{x:\overline{n}|}$ or $P(A^1_{x:\overline{n}|})$. Thus

$$P^1_{x:\overline{n}|} = P(A^1_{x:\overline{n}|}) = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$$

and if n is infinity, we obtain

$$P_x = P(A_x) = \frac{A_x}{\ddot{a}_x} = \frac{1 - d\ddot{a}_x}{\ddot{a}_x} = \frac{1}{\ddot{a}_x} - d.$$

The symbol for an n-year endowment insurance of $1 whose death benefit is payable at the end of the year of death is $P_{x:\overline{n}|}$ or $P(A_{x:\overline{n}|})$. We have

$$P_{x:\overline{n}|} = P(A_{x:\overline{n}|}) = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = \frac{1 - d\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = \frac{1}{\ddot{a}_{x:\overline{n}|}} - d.$$

If the premium of P per annum is payable $p$thly, we add the superscript (p) to P: $P^{(p)}$. For example,

$$P^{(p)}_{\;\;\underset{x:\overline{n}|}{1}} = P^{(p)}(A^{\;1}_{x:\overline{n}|}) = \frac{A^{\;1}_{x:\overline{n}|}}{\ddot{a}^{(p)}_{x:\overline{n}|}},$$

$$P^{(p)}_{x:\overline{n}|} = P^{(p)}(A_{x:\overline{n}|}) = \frac{A_{x:\overline{n}|}}{\ddot{a}^{(p)}_{x:\overline{n}|}}.$$

If the premium paying period (say m years) is shorter than the term of the insurance (say n years), we add the prefix m to P: ${}_mP$. For example,

$${}_mP^1_{x:\overline{n}|} = {}_mP(A^1_{x:\overline{n}|}) = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}},$$

and

$$ {}_mP_x^{(p)} = {}_mP^{(p)}(A_x) = \frac{A_x}{\ddot{a}_{x:\overline{m}|}^{(p)}}. $$

If a death benefit is payable at the moment of death, but the premium is still paid in the form of an annuity-due, we always use the longer notation with writing "—" above A: $P(\overline{A})$. For example,

$$ P(\overline{A}_x) = \frac{\overline{A}_x}{\ddot{a}_x} $$

and

$$ P^{(p)}(\overline{A}_{x:\overline{n}|}) = \frac{\overline{A}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}^{(p)}}. $$

If the premium is paid continuously, the longer notation is used with writing "—" above P: $\overline{P}(A)$. For example,

$$ {}_m\overline{P}(A^1_{x:\overline{n}|}) = \frac{A^1_{x:\overline{n}|}}{\overline{a}_{x:\overline{m}|}}, $$

$$ \overline{P}(\overline{A}_x) = \frac{\overline{A}_x}{\overline{a}_x} = \frac{1-\delta\,\overline{a}_x}{\overline{a}_x} = \frac{1}{\overline{a}_x} - \delta $$

and

$$ \overline{P}(\overline{A}_{x:\overline{n}|}) = \frac{\overline{A}_{x:\overline{n}|}}{\overline{a}_{x:\overline{n}|}} = \frac{1-\delta\,\overline{a}_{x:\overline{n}|}}{\overline{a}_{x:\overline{n}|}} = \frac{1}{\overline{a}_{x:\overline{n}|}} - \delta. $$

In general, we can define a symbol for the premium as follows. Let EPV denote the expected present value of the benefits of an insurance (e.g. A_x, $\overline{A}_{x:\overline{n}|}$, ${}_{n|}\ddot{a}_x$). Moreover, assume the premiums are paid in the form of an m-year life annuity. If the premiums are payable annually in advance, the annual premium is denoted by ${}_mP(EPV)$ and it can be obtained as

$$ {}_mP(EPV) = \frac{EPV}{\ddot{a}_{x:\overline{m}|}}. $$

If the premiums are payable $p$thly in advance, the annual premium is denoted by ${}_mP^{(p)}(EPV)$ and can be computed from

$${}_mP^{(p)}(EPV) = \frac{EPV}{\ddot{a}^{(p)}_{x:\overline{m}|}}.$$

If the premium is payable continuously, the rate of payment per annum is denoted by ${}_m\overline{P}\,(EPV)$ and can be determined from

$${}_m\overline{P}\,(EPV) = \frac{EPV}{\overline{a}_{x:\overline{m}|}}.$$

Note that in the case of annuities it does not make sense to define a payment period coinciding with the term of the insurance. For example, if the life annuity pays $1 in advance for n years and the annual premiums are also paid in the same interval, we get

$$P\,\ddot{a}_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|}$$

and therefore,

$$P = \$1.$$

That means the premium payments and the benefit payments are of equal amount and take place at the same time, so the whole insurance is meaningless.

However, we can consider a situation, where the premium of a deferred annuity is paid in installments before the benefit payment period starts. For example, if an n year annuity-due of $1 per annum is deferred for ℓ years, then the annual premium payable in advance for m years ($m \le \ell$) is denoted by ${}_mP({}_{\ell|}\ddot{a}_{x:\overline{n}|})$, and we have

$${}_mP({}_{\ell|}\ddot{a}_{x:\overline{n}|}) = \frac{{}_{\ell|}\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}}.$$

If $m = \ell$, we get

$${}_mP({}_{m|}\ddot{a}_{x:\overline{n}|}) = \frac{\ddot{a}_{x:\overline{m+n}|} - \ddot{a}_{x:\overline{m}|}}{\ddot{a}_{x:\overline{m}|}} = \frac{\ddot{a}_{x:\overline{m+n}|}}{\ddot{a}_{x:\overline{m}|}} - 1,$$

and if n is infinity, we obtain

$$_mP(m \mid \ddot{a}_x) = \frac{\ddot{a}_x}{\ddot{a}_{x:\overline{m}|}} - 1.$$

Of course, the premium payments can have other structures, too. For example, the installments can change during the premium payment period. However, using (1) we can always find what the necessary net premium is.

EXAMPLE 1.2. A person aged 35 takes out a 20 year pure endowment insurance of \$30,000. Find the net annual premium at a 6% annual rate of interest.

Solution: The annual premium is $3000\, P_{35:\overset{1}{\overline{20}|}}$, where

$$\begin{aligned} P_{35:\overset{1}{\overline{20}|}} &= \frac{A_{35:\overset{1}{\overline{20}|}}}{\ddot{a}_{35:\overline{20}|}} \\ &= \frac{D_{55}}{N_{35} - N_{55}} \\ &= \frac{3505.37}{188663.76 - 43031.29} \\ &= 0.0240700. \end{aligned}$$

So the premium is $30000 \times 0.0240700 = \$722.10$ per annum.

EXAMPLE 1.3. The payments of a life annuity of \$800 per month start at the age of 65. How much is the monthly premium if it is payable between the ages of 30 and 65? Use a 6% annual rate of interest.

Solution: The monthly premium is

$$\frac{800 \times 12\ {}_{35}P^{(12)}\,({}_{35|}\ddot{a}_{30}^{(12)})}{12} = 800\ {}_{35}P^{(12)}\,({}_{35|}\ddot{a}_{30}^{(12)}).$$

Now,

$$\begin{aligned} {}_{35}P^{(12)}\,({}_{35|}\ddot{a}_{30}^{(12)}) &= \frac{{}_{35|}\ddot{a}_{30}^{(12)}}{\ddot{a}_{30:\overline{35}|}^{(12)}} \\ &= \frac{\ddot{a}_{30}^{(12)} - \ddot{a}_{30:\overline{35}|}^{(12)}}{\ddot{a}_{30:\overline{35}|}^{(12)}} \\ &= \frac{\ddot{a}_{30}^{(12)}}{\ddot{a}_{30:\overline{35}|}^{(12)}} - 1. \end{aligned}$$

We can write

$$\ddot{a}_{30}^{(12)} \approx \ddot{a}_{30} - \frac{12-1}{2 \times 12}$$
$$= 15.85612 - \frac{11}{24}$$
$$= 15.39779$$

and

$$\ddot{a}_{30:\overline{35}|}^{(12)} \approx \ddot{a}_{30:\overline{35}|} - \frac{12-1}{2 \times 12}(1 - {}_{35}E_{30})$$
$$= \frac{N_{30} - N_{65}}{D_{30}} - \frac{11}{24}\left(1 - \frac{D_{65}}{D_{30}}\right)$$
$$= \frac{262305.71 - 16890.50}{16542.86} - \frac{11}{24}\left(1 - \frac{1706.64}{16542.86}\right)$$
$$= 14.42406.$$

Thus,

$${}_{35}P^{(12)}(35 \mid \ddot{a}_{30}^{(12)}) = \frac{15.39779}{14.42406} - 1$$
$$= 0.06751,$$

and hence the monthly premium is $800 \times 0.06751 = \$54.01$.

EXAMPLE 1.4. A whole life insurance of \$5000 is issued to a person aged 60. The annual premium is reduced by one third after the age of 65. Find the annual premiums based on a 6% annual rate of interest.

Solution: Denoting the premium payable in the first year by P, the expected present value of the premiums is

$$P\,\ddot{a}_{60:\overline{5}|} + \frac{2}{3} P\; {}_{5|}\ddot{a}_{60} = P(\ddot{a}_{60:\overline{5}|} + \frac{2}{3}\, {}_{5|}\ddot{a}_{60}),$$

and the expected present value of the benefits is

$$5000\, A_{60}.$$

Equating the two present values, we get

$$P(\ddot{a}_{60:\overline{5}|} + \frac{2}{3}\, {}_{5|}\ddot{a}_{60}) = 5000\, A_{60}$$

and hence,

$$P = 5000\,\frac{A_{60}}{\ddot{a}_{60:\overline{5}|} + \left(\frac{2}{3}\right) {}_{5|}\ddot{a}_{60}}.$$

Now,

$$A_{60} = 0.3691310,$$

$$\ddot{a}_{60:\overline{5}|} = \frac{N_{60} - N_{65}}{D_{60}} = \frac{27664.55 - 16890.50}{2482.16} = 4.34059,$$

and

$$5\,|\,\ddot{a}_{60} = \frac{N_{65}}{D_{60}} = \frac{16890.50}{2482.16} = 6.80476.$$

Thus, the annual premium is

$$5000\,\frac{0.3691310}{4.34059 + \left(\frac{2}{3}\right) 6.80476} = \$207.91$$

in the first five years, and $\left(\frac{2}{3}\right) 207.91 = \$138.61$ afterwards.

All the insurances discussed so far contained benefits whose payment depended on death or survival of the insured. However, it is possible that the payment of the benefit is guaranteed and only the premium payments are contingent on survival. Next we study this type of insurance. It is called a "term certain" insurance. So consider the following arrangement.

A person aged x makes regular payments for n years while he/she is alive. In return, the insurance makes a payment of $1 at the end of year n, whether the insured is alive or not.

For example, assume the premium is paid in the form of an annuity-due. Denoting the annual premium by P, we get

$$P\,\ddot{a}_{x:\overline{n}|} = v^n$$

i.e.

$$P = \frac{v^n}{\ddot{a}_{x:\overline{n}|}}.$$

Note that if $1 is only payable on survival; that is, we have a pure endowment, the annual premium, say P^*, is

$$P^* = \frac{{}_nE_x}{\ddot{a}_{x:\overline{n|}}} = {}_np_x \frac{v^n}{\ddot{a}_{x:\overline{n|}}} = {}_np_x\, P.$$

Therefore, $P > P^*$, which is reasonable since a price has to be paid for the guarantee the benefit is paid even if the insured dies before the end of year n.

If the premium is paid continuously, we get

$$P = \frac{v^n}{\overline{a}_{x:\overline{n|}}}.$$

EXAMPLE 1.5. An insurance provides a guaranteed benefit of \$12,000, 20 years after it is taken out by a person aged 40. Find the annual premium based on a 6% annual rate of interest.

Solution: Denoting the annual premium by P, we have

$$P\, \ddot{a}_{40:\overline{20|}} = 12000\, v^{20}.$$

Now

$$v^{20} = 0.31180$$

and

$$\ddot{a}_{40:\overline{20|}} = \frac{N_{40} - N_{60}}{D_{40}} = \frac{134156.39 - 27664.55}{9054.46} = 11.7613.$$

Thus the annual premium is

$$P = 12000\, \frac{0.31180}{11.7613} = \$318.13.$$

Note that the annual premium of the corresponding pure endowment is

$$\begin{aligned} P^* &= 12000 \times \frac{{}_{20}E_{40}}{\ddot{a}_{40:\overline{20|}}} \\ &= 12000 \times \frac{D_{60}}{D_{40}} \times \frac{1}{\ddot{a}_{40:\overline{20|}}} \\ &= 12000 \times \frac{2482.16}{9054.46} \times \frac{1}{11.7613} \\ &= \$279.70 \end{aligned}$$

which is less than \$318.13.

It is also possible that the benefit of an insurance depends explicitly on the premium as the following example shows.

EXAMPLE 1.6. A person aged 40 takes out an insurance with a survival benefit of \$15000 payable at the age of 60. The premium is paid annually and if death occurs before the age of 60, all the premiums received by the insurance company are returned without interest at the end of the year of death. Find the annual premium at a 6% annual rate of interest.

Solution: Let P be the annual premium. Then, the present value of the premiums is $P\,\ddot{a}_{40:\overline{20|}}$. If death occurs in the first year, P is returned at the end of the first year. If death occurs in the second year, $2P$ is returned at the end of the second year, etc. If death occurs in year 20, an amount of $20P$ is returned at the end of year 20. Finally, if the insured survives 20 years, he/she receives \$15000. So the present value of the benefit is $P(IA)^{1}_{40:\overline{20|}} + 15000\,A_{40:\overset{1}{\overline{20|}}}$. Thus,

$$P\,\ddot{a}_{40:\overline{20|}} = P(IA)^{1}_{40:\overline{20|}} + 15000\,A_{40:\overset{1}{\overline{20|}}}$$

and

$$P = 15000\,\frac{A_{40:\overset{1}{\overline{20|}}}}{\ddot{a}_{40:\overline{20|}} - (IA)^{1}_{40:\overline{20|}}}.$$

Now,

$$A_{40:\overset{1}{\overline{20|}}} = \frac{D_{60}}{D_{40}} = \frac{2482.16}{9054.46} = 0.2741367,$$

$$\ddot{a}_{40:\overline{20|}} = \frac{N_{40} - N_{60}}{D_{40}} = \frac{134156.39 - 27664.55}{9054.46} = 11.76126,$$

and

$$(IA)^{1}_{40:\overline{20|}} = \frac{R_{40} - R_{60} - 20\,M_{60}}{D_{40}}$$

$$= \frac{37787.4414 - 13459.2908 - 20 \times 916.2423}{9054.46}$$

$$= 0.66302.$$

Hence, the annual premium is

$$P = 15000\,\frac{0.2741367}{11.76126 - 0.66302} = \$370.51.$$

Note that without the premium return option, the annual premium would be

$$P = 15000\,\frac{A^{\;\;1}_{40:\overline{20|}}}{\ddot{a}_{40:\overline{20|}}} = \$349.63.$$

Therefore, the extra premium required for the premium return option is relatively small: \$20.88. An insured is probably ready to pay this amount if it is guaranteed that the money paid to the insurance company will not be lost in the case of an early death.

The premium calculations discussed so far are all based on expected values. Since the insurance companies usually have a large number of policies, the expected value describes the average actual experience quite accurately.

However, it is also interesting to study the behavior of a small number of policies.

Assume the premium of an insurance depends on the parameter P. For example, P is a single premium or an installment of a level annual premium. Consider a person who takes out the insurance at the age of x and dies at the age of $x + t$. Then the cash flow corresponding to this particular policy consists of premium and benefit payments. Looking at the cash flow from the point of view of the insurance company, the premium payments are positive amounts and the benefit payments are negative quantities. Let us denote the present value of the cash flow at the time the insurance starts by $g(t,P)$. Then, $g(t,P)$ gives the present value of the profit of the company resulting from the policy. Obviously, the goal of the insurance company is to make profit, so it wants to achieve

$$g(t,P) > 0.$$

Since the time of death cannot be foreseen at the commencement of the insurance, we have to use the random variable $g(T_x,P)$.

In order to determine P, we may require that the policy be profitable to the insurance company with a high probability:

$$P(g(T_x,P) \geq 0)) \geq 1 - \alpha, \tag{3}$$

where α is a small positive number.

For example, consider a whole life insurance of \$1 payable at the end of the year of death on a life aged x. Assume we want to find a single premium so that the policy is profitable with probability $1 - \alpha$:

Denoting the single premium by P, we can write

$$g(t,P) = P - v^{[t]+1}.$$

So, we have to find P satisfying

$$P(P - v^{[T_x]+1} \geq 0) \geq 1 - \alpha;$$

that is,

$$P(P - v^{K_x+1} \geq 0) \geq 1 - \alpha. \tag{4}$$

This can be rewritten as

$$P(\log P \geq (K_x + 1)\log v) \geq 1 - \alpha.$$

Since $v < 1$, we have $\log v < 0$, thus we get

$$P\left(K_x \geq \frac{\log P}{\log v} - 1\right) \geq 1 - \alpha.$$

Now,

$$P(K_x \geq n) = \frac{\ell_{x+n}}{\ell_x}.$$

Therefore,

$$P(K_x \geq n) \geq 1 - \alpha$$

is satisfied if

$$\frac{\ell_{x+n}}{\ell_x} \geq 1 - \alpha.$$

Let us select the largest integer n from the mortality table for which

$$\ell_{x+n} \geq \ell_x(1 - \alpha). \tag{5}$$

Then, we must have

$$\frac{\log P}{\log v} - 1 \leq n$$

i.e.

$$\log P \geq (1+n)\log v$$

which implies

$$P \geq v^{n+1}.$$

Hence the smallest premium that can be charged is

$$P = v^{n+1}$$

where n is the largest integer satisfying (5).

EXAMPLE 1.7. Find the single premium for a whole life insurance with sum insured $2000 payable at the end of the year of death for a life aged 50 so that the policy is profitable to the company with 95% probability. Use a 6% annual rate of interest.

Solution: Using (5), we have to find the largest n for which

$$\ell_{50+n} \geq \ell_{50}\, 0.95 = 89509.00 \times 0.95 = 85033.55.$$

Since $\ell_{56} = 85634.33$ and $\ell_{57} = 84799.07$, we get $n = 6$. Now,

$$v^{n+1} = v^7 = 0.66506.$$

Hence, the premium is $P = 0.66506 \times 2000 = \$1330.12$. Note that if we determine the premium from the expected present value equation (2), we get

$$P^* = 2000\, A_{50} = 2000 \times 0.2490475 = \$498.095,$$

which is much smaller than $1330.12.

It is generally true that a premium calculation based on expected values gives a much smaller premium than the method described above. Since people are always looking for lower prices, the practical use of the latter method is rather limited.

Let us see next that if we have N policies, each sold to a person aged x, how we can determine P, so that the present value of the total profit is greater than zero with a high probability. Let us denote the future lifetime of the N insured by $(T_x)_1, (T_x)_2, \ldots, (T_x)_N$. We will assume that the random variables $(T_x)_i$ are not only identically distributed but also independent. The present value of the total profit is also a random variable. We will denote it by $X(P)$

$$X(P) = \sum_{i=1}^{N} g((T_x)_i, P). \tag{6}$$

We want to choose P so that

$$P(X(P) > 0) \geq 1 - \alpha. \tag{7}$$

Since $X(P)$ is the sum of independent, identically distributed random variables, we can use the central limit theorem to approximate $P(X(P) > 0)$, if N is large enough, say $N \geq 50$. Rewriting (7) as

$$P\left(\frac{X(P) - E(X(P))}{\sqrt{V(X(P))}} \geq \frac{0 - E(X(P))}{\sqrt{V(X(P))}}\right) \geq 1 - \alpha,$$

we have

$$P\left(Z \geq -\frac{E(X(P))}{\sqrt{V(X(P))}}\right) \geq 1 - \alpha, \tag{8}$$

where Z has the standard normal distribution. Hence, (8) is equivalent to

$$\frac{E(X(P))}{\sqrt{V(X(P))}} \geq z_{1-\alpha},$$

where $z_{1-\alpha}$ is the $100(1 - \alpha)$ percentage point of the standard normal distribution. In our examples, we will use $\alpha = 0.05$. Then, $z_{1-\alpha} = z_{0.95} = 1.645$. Therefore, P is the solution of the equation

$$E(X(P)) = z_{1-\alpha}\sqrt{V(X(P))}\ . \tag{9}$$

Formula (9) can be simplified if P is a single premium and the present value of the benefits of the insurance can be expressed as $g_1(T_x)$. Then, we get

$$g(T_x, P) = P - g_1(T_x),$$

so

$$E(X(P)) = N(P - E(g_1(T_x))) \tag{10}$$

and

$$V(X(P)) = NV(g_1(T_x)). \tag{11}$$

Therefore, (9) implies

$$N(P - E(g_1(T_x)) = z_{1-\alpha}\sqrt{NV(g_1(T_x))}$$

and thus,

$$P = E(g_1(T_x)) + \frac{z_{1-\alpha}}{\sqrt{N}}\sqrt{V(g_1(T_x))}\,. \tag{12}$$

Note that as N gets larger, the second term on the right hand side of (12) goes to zero, so the premium approaches the present value of the benefits.

EXAMPLE 1.8. Find the net single premium for the whole life insurance, given in Example 3.1 of Section 3.3, so that the insurance company makes a profit with 95% probability, if the insurance is sold to a group of

a) 50 people
b) 100 people
c) 1000 people
d) 10,000 people

Solution: In Example 3.1, we found

$$E(g_1(T_x)) = 373.57$$

and

$$\sqrt{V(g_1(T_x))} = 271.38.$$

Using (12), we get

$$P = 373.57 + \frac{1.645}{\sqrt{N}}\, 271.38.$$

So, we obtain

a) $P = \$436.70$
b) $P = \$418.21$
c) $P = \$387.69$
d) $P = \$378.03$

It can be seen that the premiums tend to \$343.57 as the number of policies increases.

If the premium is paid in the form of an annuity, the formula for P is usually not very handy. If the present value of the premiums is $P\, g_2(T_x)$ and the present value of the benefits is $g_1(T_x)$, then

$$g(T_x,P) = P\, g_2(T_x) - g_1(T_x).$$

Hence, we have

$$E(X(P)) = NE(P\, g_2(T_x) - g_1(T_x)) \tag{13}$$

which can be calculated easily. However, we have

$$V(X(P)) = N\, V(P\, g_2(T_x) - g_1(T_x)), \tag{14}$$

which is often rather complicated to determine. However, in certain cases, $V(X(P))$ can be calculated easily. For example, consider an n-year endowment insurance of \$1.

If the death benefit is paid at the end of the year of death and the premium payments form an n-year annuity-due, then $g_1(T_x) = h_1(K_x)$, where

$$h_1(k) = \begin{cases} v^{k+1}, & \text{if } k < n \\ v^n, & \text{if } n \le k \end{cases}$$

(see (1) of Section 3.4), and $g_2(T_x) = h_2(K_x)$, where

$$h_2(k) = \begin{cases} \dfrac{1 - v^{k+1}}{d}, & \text{if } k < n \\[2ex] \dfrac{1 - v^n}{d}, & \text{if } n \le k \end{cases}$$

(see (8) of Section 3.5). Thus,

$$h_2(k) = \frac{1}{d}\,(1 - h_1(k)),$$

and

$$\begin{aligned} g(T_x,P) &= P\, h_2(K_x) - h_1(K_x) \\ &= \frac{P}{d}\,(1 - h_1(K_x)) - h_1(K_x) \\ &= \frac{P}{d} - \left(\frac{P}{d} + 1\right) h_1(K_x). \end{aligned}$$

Hence,

$$V(g(T_x,P)) = \left(\frac{P}{d} + 1\right)^2 V(h_1(K_x)).$$

Now $h_1(K_x)$ is the present value of the cash flow of an n-year endowment insurance. Thus,

$$V(h_1(K_x)) = {}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2$$

(see (7) of Section 3.4). Therefore,

$$V(g(T_x,P)) = \left(\frac{P}{d} + 1\right)^2 ({}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2)$$

and so

$$V(X(P)) = N\left(\frac{P}{d} + 1\right)^2 ({}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2).$$

On the other hand,

$$E(g(T_x,P)) = P\,\ddot{a}_{x:\overline{n}|} - A_{x:\overline{n}|}$$

and hence,

$$E(X(P)) = N(P\,\ddot{a}_{x:\overline{n}|} - A_{x:\overline{n}|}).$$

Now from (9), we get

$$N(P\,\ddot{a}_{x:\overline{n}|} - A_{x:\overline{n}|}) = z_{1-\alpha}\sqrt{N}\left(\frac{P}{d} + 1\right)\sqrt{{}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2}\,.$$

Rearranging the terms, we get

$$P\left(\ddot{a}_{x:\overline{n}|} - \frac{z_{1-\alpha}}{d\sqrt{N}}\sqrt{{}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2}\right) = A_{x:\overline{n}|} + \frac{z_{1-\alpha}}{\sqrt{N}}\sqrt{{}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2}\,,$$

from which we obtain

$$P = \frac{A_{x:\overline{n}|} + \dfrac{z_{1-\alpha}}{\sqrt{N}}\sqrt{{}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2}}{\ddot{a}_{x:\overline{n}|} - \dfrac{z_{1-\alpha}}{d\sqrt{N}}\sqrt{{}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2}}$$

$$= \frac{A_{x:\overline{n}|} + \dfrac{z_{1-\alpha}}{\sqrt{N}}\sqrt{V(A_{x:\overline{n}|})}}{\ddot{a}_{x:\overline{n}|} - \dfrac{z_{1-\alpha}}{\sqrt{N}}\sqrt{V(\ddot{a}_{x:\overline{n}|})}}\,. \qquad (15)$$

If n is infinity, the insurance becomes a whole life insurance and we get

$$P = \frac{A_x + \frac{z_{1-\alpha}}{\sqrt{N}} \sqrt{{}^2A_x - A_x^2}}{\ddot{a}_x - \frac{z_{1-\alpha}}{d\sqrt{N}} \sqrt{{}^2A_x - A_x^2}}$$

$$= \frac{A_x + \frac{z_{1-\alpha}}{\sqrt{N}} \sqrt{V(A_x)}}{\ddot{a}_x - \frac{z_{1-\alpha}}{\sqrt{N}} \sqrt{V(\ddot{a}_x)}}. \tag{16}$$

If the death benefit of the endowment is paid at the moment of death and the premium is paid continuously at a rate of P per annum, then following the lines of the above derivation and using (8) of Section 3.4 and (70) of Section 3.5, we get

$$E(X(P)) = N(P\,\overline{a}_{x:\overline{n}|} - \overline{A}_{x:\overline{n}|}),$$

$$V(X(P)) = N\left(\frac{P}{\delta} + 1\right)^2 ({}^2\overline{A}_{x:\overline{n}|} - (\overline{A}_{x:\overline{n}|})^2),$$

and

$$P = \frac{\overline{A}_{x:\overline{n}|} + \frac{z_{1-\alpha}}{\sqrt{N}} \sqrt{{}^2\overline{A}_{x:\overline{n}|} - (\overline{A}_{x:\overline{n}|})^2}}{\overline{a}_{x:\overline{n}|} - \frac{z}{\delta\sqrt{N}} \sqrt{{}^2\overline{A}_{x:\overline{n}|} - (\overline{A}_{x:\overline{n}|})^2}}$$

$$= \frac{\overline{A}_{x:\overline{n}|} + \frac{z_{1-\alpha}}{\sqrt{N}} \sqrt{V(\overline{A}_{x:\overline{n}|})}}{\overline{a}_{x:\overline{n}|} - \frac{z_{1-\alpha}}{\sqrt{N}} \sqrt{V(\overline{a}_{x:\overline{n}|})}}. \tag{17}$$

If n is infinity, then we get

$$P = \frac{\overline{A}_x + \frac{z_{1-\alpha}}{\sqrt{N}} \sqrt{{}^2\overline{A}_x - \overline{A}^2}}{\overline{a}_x - \frac{z_{1-\alpha}}{\delta\sqrt{N}} \sqrt{{}^2\overline{A}_x - \overline{A}^2}}$$

$$= \frac{\overline{A}_x + \frac{z_{1-\alpha}}{\sqrt{N}} \sqrt{V(\overline{A}_x)}}{\overline{a}_x - \frac{z_{1-\alpha}}{\sqrt{N}} \sqrt{V(\overline{a}_x)}}. \tag{18}$$

EXAMPLE 1.9. Determine the net annual premium for the endowment insurance given in part (a) of Example 4.1 of Section 3.4, so that the insurance company makes a profit with 95% probability on a group of

a) 50 people
b) 100 people
c) 1000 people
d) 10,000 people.

Solution: Since the annual rate of interest is 6%, the discount rate is $d = 0.056604$. From Example 4.1 of Section 3.4, we know that

$$A_{40:\overline{20|}} = 0.33427$$

and

$$V(A_{40:\overline{20|}}) = 0.0072090.$$

We also need to find $\ddot{a}_{40:\overline{20|}}$. Since $A_{40:\overline{20|}}$ is already given, we can obtain it from

$$\begin{aligned} \ddot{a}_{40:\overline{20|}} &= \frac{1}{d}\,(1 - A_{40:\overline{20|}}) \\ &= \frac{1}{0.056604}\,(1 - 0.33427) \\ &= 11.76118, \end{aligned}$$

or we can compute it directly as

$$\begin{aligned} \ddot{a}_{40:\overline{20|}} &= \frac{N_{40} - N_{60}}{D_{40}} \\ &= \frac{134156.39 - 27664.55}{9054.46} \\ &= 11.76126. \end{aligned}$$

Using (15), we get

$$P = 5000\,\frac{0.33427 + \frac{1.645}{\sqrt{N}}\sqrt{0.0072090}}{11.76126 - \frac{1.645}{0.056604\,\sqrt{N}}\sqrt{0.0072090}}$$

$$= \frac{1671.35 + \dfrac{698.35}{\sqrt{N}}}{11.76126 - \dfrac{2.46750}{\sqrt{N}}}.$$

Therefore, we obtain the following results

a) $P = \$155.11$
b) $P = \$151.22$
c) $P = \$144.95$
d) $P = \$143.00$.

We can see that as the number of policies gets larger, the premiums approach $5000\dfrac{A_{40:\overline{20|}}}{\ddot{a}_{40:\overline{20|}}} = \$142.11.$

PROBLEMS

1.1. A whole life insurance is issued to a life aged 50. The death benefit payable at the end of the year of death is \$5000, if death occurs in the first 10 years and \$2000 afterwards. Obtain the single premium at the age of 50, based on a 6% annual rate of interest.

1.2. The annual payment of a 15 year annuity immediate is \$800 in the first 5 years and \$1400 for the rest of the term. Find the single premium of the annuity at the age of 30 using a 6% annual rate of interest.

1.3. An insurance issued to a life aged 40 pays a death benefit of \$5000 at the moment of death, if death occurs within 20 years and \$7000 on survival to age 60. Determine the single premium based on a 6% annual interest rate.

1.4. Based on a 6% annual rate of interest, obtain

a) $P_{40:\overline{25|}}^{\;\;\,1}$
b) $P_{30:\overline{10|}}^{1}$
c) $P_{50:\overline{15|}}$
d) P_{60}
e) $P^{(12)}{}_{25:\overline{10|}}^{\;\;\,1}$
f) $P_{35:\overline{5|}}^{(4)}$
g) ${}_{10}P_{20}$
h) ${}_{20}P_{25:\overline{30|}}$
i) $P(\bar{A}_{50:\overline{20|}})$

j) $P^{(4)}(\bar{A}^{\,1}_{40:\overline{10}|})$

k) $\bar{P}(A_{20})$

l) $\bar{P}(\bar{A}_{50:\overline{20}|})$

m) ${}_{10}P({}_{10|}\ddot{a}_{30})$

n) $P^{(4)}((IA)_{40:\overline{10}|})$.

1.5. A whole life insurance is issued to a life aged 40, with a death benefit of \$3000, payable at the moment of death. Based on a 6% annual rate of interest, determine the annual premium if it is paid

a) yearly in advance

b) continuously for the whole term of the insurance.

1.6. A 30 year endowment insurance of \$5000, whose death benefit is payable at the end of the year of death, is issued to a life aged 20. Determine the monthly premium based on a 6% annual rate of interest.

1.7. A 15 year term insurance with a \$3000 death benefit payable at the end of the year of death is issued to a life aged 50. The premium is payable continuously for a term of 5 years. Find the annual rate of the premium payment based on a 6% annual interest rate.

1.8. A continuous life annuity of \$2000 per annum deferred 15 years is purchased for a life aged 50 by monthly installments payable for 5 years. Determine the monthly premium using a 6% annual interest rate.

1.9. The monthly premium for a 20 year term insurance of \$9000 is increased by 20% after the first 10 years. Find the monthly premium, if the insurance is issued to a life aged 50. Use a 6% annual rate of interest.

1.10. A term certain insurance of \$5000 is taken out at the age of 50 for a term of 10 years. Based on a 6% annual rate of interest, find the annual premium if it is payable

a) over the full term of the insurance

b) for the first 5 years.

1.11. A 15 year pure endowment insurance of \$5000 is issued to a life aged 30. If the insured dies before the age of 45, the sum of the annual premiums received by the insurance company are returned without interest at the end of the year of death. Determine the annual premium based on a 6% annual interest rate.

1.12. A life annuity of \$6000 per annum, payable monthly in advance with a deferment period of 10 years, is purchased for a life aged 50 by a single premium. Based on a 6% annual rate of interest,

a) obtain the single premium

b) obtain the single premium under the condition it is returned without interest at the end of the year of death, if death occurs before the age of 60.

1.13. A whole life insurance on a life aged 40 pays a death benefit of \$4000 at the end of the year of death. Based on a 6% annual rate of interest find the single premium so that the policy is profitable to the company with 95% probability.

1.14. Determine the net single premium for the temporary insurance, given in Problem 3.10 of Section 3.3, so that the insurance company makes a profit with 95% probability on a group of

a) 60 people

b) 150 people

c) 2000 people

d) 8000 people

1.15. Find the net annual premium for the pure endowment insurance, given in Problem 3.3 of Section 3.3, so that the insurance company makes a profit with 95% probability on a group of

a) 70 people

b) 300 people

c) 5000 people

d) 10,000 people

1.16. Assume the premium for the endowment insurance, given in Problem 4.4 of Section 3.4, is payable continuously. Determine the constant premium payment rate per annum so that the insurance company makes a profit with 95% probability on a group of

a) 50 people

b) 120 people

c) 1000 people

d) 9000 people

4.2. Gross Premiums

When an insurance company determines the premium for an insurance, it has to take into account expenses as well. By expense, we mean the costs incurred by the insurance company to run the business; for example, salaries paid to the staff, commissions paid to the agents, rent paid for the offices, postage paid for mailing a check to the policyholder, etc.

Expenses can be categorized in two ways. One possibility is to focus on when the expenses are incurred. Expenses incurred at the time the insurance policy is issued are called initial expenses. The commissions paid to the agents are typical examples of them. Other expenses, incurred regularly while the policy is in force, are called renewal expenses. For example, administration costs are part of the renewal expenses. Since the terminology is ambiguous, it is always important to check whether a renewal expense is incurred at the start of the policy or not. There are other expenses as well; for example, claims expenses which are incurred when a benefit payment is made.

On the other hand, we can examine the factors determining the amount of the expenses. There are expenses related to the premium, to the benefit, or to neither of them. For example, commissions can be determined based on the premium or the benefit. On the other hand, the rent for an office does not depend on either of these two factors. In many cases, the expenses are given as a percentage of the premium or the benefit.

As we have already mentioned at the beginning of this chapter, if the expenses are included in the premium, we are talking about a gross (or official) premium.

The basic equation from which the gross premium can be obtained is

$$\begin{aligned}&\text{Expected present value of the gross premiums}\\&\qquad = \text{Expected present value of the benefits}\\&\qquad\qquad + \text{Expected present value of the expenses.}\end{aligned} \tag{1}$$

If we compare this equation with (1) of Section 4.1, we can also write

$$\begin{aligned}&\text{Expected present value of the gross premiums}\\&\qquad = \text{Expected present value of the net premiums}\\&\qquad\qquad + \text{Expected present value of the expenses.}\end{aligned} \tag{2}$$

In the previous section, we denoted the net premium by P. The symbol generally used for the gross premium is P''. Their difference, $P'' - P$ is called the expense loading.

EXAMPLE 2.1. A 10 year endowment insurance of $\$6000$ is sold to a person aged 50. There are initial expenses of $\$200$, renewal expenses of 0.2% of the sum insured, incurred at the beginning of each year including the first, and claims expenses of $\$10$. Find the gross single premium at a 6% annual rate of interest. What is the expense loading of the premium?

Solution: Let P'' be the gross single premium. Then, we have

$$\begin{aligned}P'' &= 6000\, A_{50:\overline{10}|} + 200 + 0.002 \times 6000\, \ddot{a}_{50:\overline{10}|} + 10\, A_{50:\overline{10}|}\\ &= 6010\, A_{50:\overline{10}|} + 0.002 \times 6000\, \ddot{a}_{50:\overline{10}|} + 200.\end{aligned}$$

Now,

$$\begin{aligned} A_{50:\overline{10|}} &= \frac{M_{50} - M_{60} + D_{60}}{D_{50}} \\ &= \frac{1210.1957 - 916.2423 + 2482.16}{4859.30} \\ &= 0.571299 \end{aligned}$$

and

$$\begin{aligned} \ddot{a}_{50:\overline{10|}} &= \frac{N_{50} - N_{60}}{D_{50}} \\ &= \frac{64467.45 - 27664.55}{4859.30} \\ &= 7.573704. \end{aligned}$$

Thus, the gross single premium is

$$\begin{aligned} P'' &= 6010 \times 0.571299 + 12 \times 7.573704 + 200 \\ &= \$3724.39. \end{aligned}$$

In order to find the expense loading, we have to determine the net premium P first:

$$P = 6000\, A_{50:\overline{10|}} = 6000 \times 0.571299 = \$3427.79.$$

Therefore, the expense loading is $\$3724.39 - \$3427.79 = \$296.60$.

EXAMPLE 2.2. Consider a monthly annuity of $\$800$ per month, payable monthly in advance from the age of 65, issued to a life aged 40. The premium is payable monthly in advance between the ages of 40 and 65. The initial expense is 30% of the monthly benefit, and there are renewal expenses of 4% of each premium excluding the first. Find the gross monthly premium using a 6% annual rate of interest. Also, find the expense loading of the monthly premium.

Solution: Let P'' denote the gross monthly premium. Then, we have

$$12P''\, \ddot{a}^{(12)}_{40:\overline{25|}} = 12 \times 800\; {}_{25|}\ddot{a}^{(12)}_{40} + 0.3 \times 800 + 12 \times 0.04\, P''\, \ddot{a}^{(12)}_{40:\overline{25|}} - 0.04\, P''.$$

Note that $0.04P''$ had to be subtracted from $12 \times 0.04\, P''\, \ddot{a}^{(12)}_{40:\overline{25|}}$, since there is no renewal expense related to the first premium. We can express P'' from the equation as

$$P'' = \frac{9600\; {}_{25|}\ddot{a}^{(12)}_{40} + 240}{12 \times 0.96\, \ddot{a}^{(12)}_{40:\overline{25|}} + 0.04}.$$

Now,

$$
\begin{aligned}
\ddot{a}^{(12)}_{40:\overline{25}|} &= \ddot{a}_{40:\overline{25}|} - \frac{12-1}{2\times 12}(1 - {}_{25}E_{40})\\
&= \frac{N_{40} - N_{65}}{D_{40}} - \frac{11}{24}\left(1 - \frac{D_{65}}{D_{40}}\right)\\
&= \frac{134156.39 - 16890.50}{9054.46} - \frac{11}{24}\left(1 - \frac{1706.64}{9054.46}\right)\\
&= 12.57923,
\end{aligned}
$$

and

$$
\begin{aligned}
{}_{25|}\ddot{a}^{(12)}_{40} &= {}_{25|}\ddot{a}_{40} - \frac{12-1}{2\times 12}\,{}_{25}E_{40}\\
&= \frac{N_{65}}{D_{40}} - \frac{11}{24}\cdot\frac{D_{65}}{D_{40}}\\
&= \frac{16890.50}{9054.46} - \frac{11}{24}\cdot\frac{1706.64}{9054.46}\\
&= 1.77904.
\end{aligned}
$$

Thus, the gross monthly premium is

$$P'' = \frac{9600 \times 1.77904 + 240}{11.52 \times 12.57923 + 0.04} = \$119.48.$$

The net monthly premium is

$$P = \frac{12 \times 800\ {}_{25|}\ddot{a}^{(12)}_{40}}{12\ \ddot{a}^{(12)}_{40:\overline{25}|}} = \frac{800 \times 1.77904}{12.57923} = \$113.14.$$

Thus, the expense loading of the monthly premium is $119.48 - 113.14 = \$6.34$.

The following theorem gives the relationship between net and gross premiums assuming a special expense structure. This expense structure can be applied in many situations.

THEOREM 2.1. *Consider an insurance whose premiums are payable for n years in the form of a yearly annuity-due (n is a positive integer or infinity). Renewal expenses proportional to the premium (say k times the annual premium, $0 \le k \le 1$) and other renewal expenses of constant amount c are incurred at the beginning of each year, while the premium is being paid. In addition to that, there is an initial expense of I. Let P be the net annual premium and P'' the gross annual premium. Then, we have*

$$P'' = \frac{1}{1-k}\left(P + c + \frac{I}{\ddot{a}_{x:\overline{n}|}}\right), \ \textit{if } n \textit{ is finite} \tag{3}$$

and

$$P'' = \frac{1}{1-k}\left(P + c + \frac{I}{\ddot{a}_{x}}\right), \ \textit{if } n \textit{ is infinity.} \tag{4}$$

Proof: From (2), we have

$$P''\,\ddot{a}_{x:\overline{n}|} = P\,\ddot{a}_{x:\overline{n}|} + k \cdot P''\,\ddot{a}_{x:\overline{n}|} + c\,\ddot{a}_{x:\overline{n}|} + I,$$

so

$$(1-k)P'' = P + c + \frac{I}{\ddot{a}_{x:\overline{n}|}}$$

and dividing both sides by $(1-k)$, we obtain (3). Taking the limit on the right hand side of (3) as n goes to infinity, we get (4). ■

In practical applications, c and I are often expressed in percentages of the benefits.

Let us interpret formula (3). Every time a gross annual premium P'' is paid, one part of it, $kP'' + c$ covers the renewal expenses. So we are left with

$$P'' - (kP'' + c) = (1-k)P'' - c = P + \frac{I}{\ddot{a}_{x:\overline{n}|}}\ .$$

The term P is the required net annual premium. What remains is $\frac{I}{\ddot{a}_{x:\overline{n}|}}$, which can be interpreted as the annual installment of a series of payments making up for the initial expense of I. Indeed, the present value of these payments is $\frac{I}{\ddot{a}_{x:\overline{n}|}}\,\ddot{a}_{x:\overline{n}|} = I$. In summary, each annual premium payment fully covers the net premium and the renewal expenses of the respective year and contains a part from which the initial expense is to be recovered over the time.

EXAMPLE 2.3. A whole life insurance of \$9000, whose premium is payable yearly in advance, is taken out at the age of 40. Expenses to be allowed for are the following: initial expenses of \$100, renewal expenses of 5% of the annual premium, and 0.2% of the sum insured. The renewal expenses are incurred every year. Find the gross annual premium based on a 6% annual rate of interest.

Solution: Let us denote the gross annual premium by P''. Then, we have

$$P''\,\ddot{a}_{40} = 9000\,A_{40} + 0.05\,P''\,\ddot{a}_{40} + 0.002 \times 9000\,\ddot{a}_{40} + 100,$$

from which

$$P'' = \frac{9000\,A_{40} + 18\,\ddot{a}_{40} + 100}{0.95\,\ddot{a}_{40}}.$$

Now,

$$A_{40} = 0.1613242$$

and

$$\ddot{a}_{40} = 14.81661.$$

Therefore, the gross annual premium is

$$P'' = \frac{9000 \times 0.1613242 + 18 \times 14.81661 + 100}{0.95 \times 14.81661}$$
$$= \$129.20.$$

EXAMPLE 2.4. The premium for a 15 year term insurance of \$5000 is payable yearly in advance. There are initial expenses of 3% of the sum insured, renewal expenses of 4% of the annual premium, and 0.25% of the sum insured. The renewal expenses are incurred every year. The insurance is sold to a person aged 50. Based on a 6% annual rate of interest, find

a) the net annual premium

b) the gross annual premium.

Solution: a) Denoting the net annual premium by P, we get

$$P\,\ddot{a}_{50:\overline{15}|} = 5000\,A^{\,1}_{50:\overline{15}|},$$

and hence

$$P = 5000\frac{A^{\,1}_{50:\overline{15}|}}{\ddot{a}_{50:\overline{15}|}}.$$

Now,

$$A^{\,1}_{50:\overline{15}|} = \frac{M_{50} - M_{65}}{D_{50}} = \frac{1210.1957 - 750.5749}{4859.30} = 0.0945858$$

and

$$\ddot{a}_{50:\overline{15}|} = \frac{N_{50} - N_{65}}{D_{50}} = \frac{64467.45 - 16890.50}{4859.30} = 9.7909.$$

So, the net annual premium is

$$P = 5000 \frac{0.0945858}{9.7909} = \$48.30.$$

b) Let P'' denote the gross annual premium. Then, we can use (3) with

$$P = 48.30$$

$$k = 0.04$$

$$c = 0.0025 \times 5000 = 12.5$$

$$I = 0.03 \times 5000 = 150.$$

Thus, the gross annual premium is

$$P'' = \frac{1}{1 - 0.04}\left(48.30 + 12.5 + \frac{150}{9.7909}\right) = \$79.29.$$

PROBLEMS

2.1. A 20 year term insurance of \$5000 on a life aged 40 is purchased by a single premium. There are initial expenses of \$100, renewal expenses of 0.1% of the sum insured, incurred at the beginning of each year including the first, and claims expenses of \$30. Based on a 6% annual interest rate, determine

a) the net single premium
b) the gross single premium
c) the expense loading of the gross premium.

2.2. The premiums for a 15 year endowment insurance of \$4000 issued to a life aged 35 are payable quarterly in advance. There is an initial expense of 2% of the sum insured. At each premium payment time, including the first, there are renewal expenses of 0.3% of the sum insured and 5% of the monthly premium. Based on a 6% annual interest rate, find the gross monthly premium and the expense loading of it.

2.3. The premiums for a life annuity-immediate of \$2000 per quarter, deferred 10 years on a life aged 50, are payable yearly in advance

for a term of 5 years. Expenses to be allowed for are initial expenses of \$100 and renewal expenses of 3% of each premium including the first. Find the gross annual premium and the expense loading of it. Use a 6% annual rate of interest.

2.4. A 20 year pure endowment insurance of \$5000 is issued to a life aged 30. Find the gross annual premium if there are initial expenses of 0.2% of the sum insured and renewal expenses of \$10, plus 7% of the annual premium. The renewal expenses are incurred every year. Use a 6% annual rate of interest.

2.5. The premiums for a 25 year endowment insurance of \$6000 on a life aged 40 are payable yearly in advance. Expenses to be allowed for are initial expenses of \$200, renewal expenses of 0.3% of the sum insured, and 5% of the annual premium. The renewal expenses are incurred every year. Based on a 6% annual interest rate, find

a) the net annual premium

b) the gross annual premium.

CHAPTER 5

RESERVES

When an insurance company issues a policy, it determines the premium so that it can cover the expected benefit payments. As time passes by, the company receives premiums and pays out benefits. If we examine a policy some time after it is issued, we may find that the policy has already terminated because its term has expired or the insured has died. Therefore, it is of no interest to the company any more. On the other hand, if the policy is still in effect, we want to make sure all the future liabilities of the company can be met. If the future premiums will not suffice, a certain fund built up from the premiums already received has to be set aside, which makes up for the deficiency of future premiums. This amount is called the reserve of the policy. Having adequate reserves is essential for the solvency of any insurance company.

When reserves are computed based on the premiums and benefits only, they are called net premium reserves. If expenses are also taken into account, we are talking about modified reserves.

5.1. NET PREMIUM RESERVES

Consider the cash flow (CF) of an insurance (including both benefit and premium payments) issued at time t_0 to a life aged x. Let the benefit payments have positive signs and the premium payments have negative signs. Without loss of generality, we can assume that $t_0 = 0$.

The net premium of the insurance is obtained from (1) of Section 4.1, so we have

$$EPV_0(CF) = 0. \tag{1}$$

Assume we have selected a time t (where t is a positive integer) such that (benefit or premium) payments after t cannot be made unless the insured survives to t. Let us denote the cash flow before t by CF_1 and after t by CF_2. If a premium payment is made at exactly t, it is assigned to CF_2. A survival benefit payable at t is also assigned to CF_2. On the other hand, a death benefit payable at t is assigned to CF_1. We say the reserve is computed just after the death benefits due at that time are paid out, but just before the survival benefits are paid and the premiums are received. Later on, we will see why we use this convention. The prospective reserve at time t (we also say at duration t) is the expected value of CF_2 at t; that is, $EPV_t(CF_2)$. We can write

Prospective reserve at t
= Expected present value of future benefits at t
- Expected present value of future premiums at t, (2)

where by future benefits and premiums, we mean the payments belonging to CF_2.

Let us recall (38) of Section 3.1, which says that

$$EPV_t(CF_2) = ACV_t(-CF_1). \tag{3}$$

Note that when we compute the accumulated value $ACV_t(-CF_1)$, the premium payments have positive signs and the benefit payments have negative signs. The expression $ACV_t(-CF_1)$ is called the retrospective reserve. So, we can write

Retrospective reserve at t
= Accumulated value of past premiums at t
- Accumulated value of past benefits at t, (4)

where by past benefits and premiums we mean the payments belonging to CF_1. In order to compute accumulated values, we will use (37) of Section 3.1:

$$ACV_t(-CF_1) = \frac{EPV_0(-CF_1)}{{}_tE_x}. \tag{5}$$

Note that (3) can be rewritten as

Prospective reserve at t = Retrospective reserve at t. (6)

We have already pointed out in Section 3.1 that the accumulated value is not just simply the balance of an account of premium and benefit payments. What we can say is that if there is a large number of people buying the same insurance, the accumulated value is the share of one surviving insured from the total fund of money on hand. This is why the reserve is also called the policy value.

Therefore, roughly speaking, we can say that the retrospective reserve is the money the company has accumulated by time t, whereas the prospective reserve is the money needed to meet future liabilities. Hence it is no surprise they are equal if the premiums are determined correctly at the start of the insurance.

The reserve at duration t is denoted by ${}_tV$. The letter V comes from the expression policy value. The prefix t can be dropped if its value is clear from the context. If we want to emphasize that the reserve is obtained in a prospective way, we write ${}_tV^{prosp}$ and for the retrospective reserve, we use the notation ${}_tV^{retro}$. So, (6) can be written in the shorter form

$$ {}_tV^{prosp} = {}_tV^{retro}. \tag{7} $$

Since a life insurance is never sold to a dead person, the death benefit payment at duration $t = 0$ is always zero. Therefore,

$$ {}_0V^{retro} = 0, $$

which can also be written as

$$ {}_0V = 0. $$

EXAMPLE 1.1. A 15 year endowment of \$4000 is purchased at the age of 40 by a single premium. Find the expressions for the prospective and retrospective reserves at the end of each policy year. Calculate the reserves numerically at the end of year 5 and year 15. Use a 6% annual rate of interest.

Solution: The premium for the insurance is

$$ \begin{aligned} P &= 4000\, A_{40:\overline{15}|} \\ &= 4000 \frac{M_{40} - M_{55} + D_{55}}{D_{40}} \\ &= 4000 \frac{1460.7038 - 1069.6405 + 3505.37}{9054.46} \\ &= \$1721.33. \end{aligned} $$

The prospective reserve at the end of year t is

$$ \begin{aligned} {}_tV^{prosp} &= 4000\, A_{40+t:\overline{15-t}|} \\ &= 4000 \frac{M_{40+t} - M_{55} + D_{55}}{D_{40+t}} \end{aligned} $$

and using (5) we find

$$ \begin{aligned} {}_tV^{retro} &= \frac{P - 4000\, A^{\;1}_{40:\overline{t}|}}{{}_tE_{40}} \\ &= \frac{1}{{}_tE_{40}} \left(P - 4000 \frac{M_{40} - M_{40+t}}{D_{40}} \right). \end{aligned} $$

Note that on the right hand side of the last equation, we have written the term $A^{\;1}_{40:\overline{t}|}$ and not $A_{40:\overline{t}|}$. The reason is that in the first 14 years no survival benefit is paid and the reserve at the end of year 15 is computed

just before the survival benefit is payable. We can rewrite the retrospective reserve as follows.

$$
{}_tV^{retro} = 4000\left(\frac{M_{40} - M_{55} + D_{55}}{D_{40}} - \frac{M_{40} - M_{40+t}}{D_{40}}\right)\frac{D_{40}}{D_{40+t}}
$$

$$
= 4000\,\frac{M_{40+t} - M_{55} + D_{55}}{D_{40+t}}.
$$

Thus the prospective and retrospective reserves are equal and the reserve at the end of year five is

$$
\begin{aligned}
{}_5V &= 4000\,\frac{M_{45} - M_{55} + D_{55}}{D_{45}} \\
&= 4000\,\frac{1339.5427 - 1069.6405 + 3505.37}{6657.69} \\
&= \$2268.22.
\end{aligned}
$$

The reserve at the end of year 15 is

$$
{}_{15}V = 4000\,\frac{M_{55} - M_{55} + D_{55}}{D_{55}} = \$4000\,,
$$

which is reasonable since \$4000 should be available for the insured who is alive at the end of year 15.

Let us assume an n year annuity of \$1 per annum is issued to a life aged x. A single premium is paid at the beginning of the insurance. What is the reserve at duration t?

If the annuity is payable in advance, the expressions for the reserves are as follows:

$$
{}_tV^{prosp} = \ddot{a}_{x+t:\overline{n-t|}}
$$

and

$$
{}_tV^{retro} = \frac{P - \ddot{a}_{x:\overline{t|}}}{{}_tE_x},
$$

where P is the single premium:

$$
P = \ddot{a}_{x:\overline{n|}}.
$$

If the annuity is payable in arrears, we have to take into account that although the payment at time t is the benefit of year t, being a survival

benefit, it is a component in the prospective and not in the retrospective reserve. Thus,

$$ {}_tV^{prosp} = 1 + a_{x+t:\overline{n-t}|} = \ddot{a}_{x+t:\overline{n-t+1}|} $$

and

$$ {}_tV^{retro} = \frac{P - a_{x:\overline{t-1}|}}{{}_tE_x}, $$

where P is the single premium:

$$ P = a_{x:\overline{n}|}. $$

If the annuity is payable continuously, we have

$$ {}_tV^{prosp} = \bar{a}_{x+t:\overline{n-t}|}, $$

and

$$ {}_tV^{retro} = \frac{P - \bar{a}_{x:\overline{t}|}}{{}_tE_x}, $$

where P is the single premium:

$$ P = \bar{a}_{x:\overline{n}|}. $$

EXAMPLE 1.2. A 20 year annuity-immediate of \$1500 per annum is issued to a life aged 50. The premium is payable at the commencement of the insurance. Calculate both the prospective and the retrospective reserves at the end of year 8.

Solution: The prospective reserve is

$$ \begin{aligned} {}_8V^{prosp} &= 1500\, \ddot{a}_{50+8:\overline{20-8+1}|} \\ &= 1500\, \ddot{a}_{58:\overline{13}|} \\ &= 1500 \frac{N_{58} - N_{71}}{D_{58}} \\ &= 1500 \frac{33186.93 - 8477.11}{2857.67} \\ &= \$12970.26. \end{aligned} $$

The retrospective reserve is

$$\begin{aligned}
{}_8V^{retro} &= 1500\frac{P - a_{50:\overline{8-1}|}}{{}_8E_{50}} \\
&= 1500\frac{a_{50:\overline{20}|} - a_{50:\overline{7}|}}{{}_8E_{50}} \\
&= 1500\,\frac{D_{50}}{D_{58}}\left(\frac{N_{51} - N_{71}}{D_{50}} - \frac{N_{51} - N_{58}}{D_{50}}\right) \\
&= 1500\frac{N_{58} - N_{71}}{D_{58}} \\
&= \$12970.26
\end{aligned}$$

coinciding with the prospective reserve.

When we defined what we meant by prospective and retrospective reserve at time t, we said that t must satisfy the condition that no payments can be made after t, unless the insured has survived to that time. However, we have already seen insurances where this condition is not even satisfied for integer t's. The family income benefit, the term certain insurance and the life annuity, with guaranteed payments for a certain time, all belong to this category. What can we do about them? Let us examine the benefit payments after death a little closer. When the insured dies, the randomness is removed from the future cash flow, since the exact value of the future lifetime random variable T_x, unknown at the beginning of the insurance, reveals itself. So from this time on, the insurance is transformed into a financial transaction whose future cash flow is determined completely. Therefore, the insurance company can find its exact present value at the time of death or at any time after that, for example, at the end of the year of death. If the insurance company sets aside this amount, it can be sure that all the future liabilities related to the policy can be met. So, from the company's point of view, the future cash flow can be replaced by a single death payment whose amount is the present value of this cash flow. Then, this cash flow does not have to be taken into consideration any more when the reserve of the insurance is computed. We say the company has capitalized the future benefits payable for policies which have become claims.

Next, we examine some special cases where this problem emerges.

First, consider an n year annuity-immediate of \$1 per annum with guaranteed payments in the first m years. The benefits of this insurance can be reinterpreted in the following way. It contains a survival benefit of \$1 payable at time $t = k$ $(k = 1,2,\dots,n)$. Moreover, if the insured dies in year k, where $k \le m$, a series of $m - k + 1$ annual payments of \$1 each are started at time $t = k$. The present value of this annuity-certain, at time $t = k$, is $\ddot{a}_{\overline{m-k+1}|}$. Therefore, this insurance is equivalent to an n year insurance with a \$1 survival benefit, payable at the end of each year, and a death benefit of $\ddot{a}_{\overline{m-k+1}|}$, payable at time $t = k$, if death occurs in year k $(1 \le k \le m)$.

We already know that the single premium for the insurance is

$$P = a_{\overline{m}|} + {}_{m|}a_{x:\overline{n-m}|}\,.$$

In order to find the reserves, we have to distinguish between two cases: $t \leq m$ or $t > m$. The prospective reserve is

$${}_tV^{prosp} = \begin{cases} 1 + a_{\overline{m-t}|} + {}_{m-t|}a_{x+t:\overline{n-m}|} = \ddot{a}_{\overline{m-t+1}|} + {}_{m-t|}a_{x+t:\overline{n-m}|}, & \text{if } t \leq m \\ 1 + a_{x+t:\overline{n-t}|} = \ddot{a}_{x+t:\overline{n-t+1}|}, & \text{if } t > m. \end{cases}$$

The retrospective reserve is

$${}_tV^{retro} = \begin{cases} \dfrac{P - \left(a_{x:\overline{t-1}|} + \sum\limits_{k=1}^{t} \dfrac{C_{x+k-1}}{D_x}\, \ddot{a}_{\overline{m-k+1}|} \right)}{{}_tE_x} & \text{if } t \leq m \\[2ex] \dfrac{P - \left(a_{x:\overline{t-1}|} + \sum\limits_{k=1}^{m} \dfrac{C_{x+k-1}}{D_x}\, \ddot{a}_{\overline{m-k+1}|} \right)}{{}_tE_x} & \text{if } t > m. \end{cases}$$

Next, let us focus on an n year family income benefit of \$1 per annum. If the insured dies in year k $(1 \leq k \leq n)$, a series of $n - k + 1$ annual payments of \$1 each are started at time $t = k$. The present value of this annuity-certain at $t = k$ is $\ddot{a}_{\overline{n-k+1}|}$. Thus, the insurance is equivalent to an n year insurance, with a death benefit of $\ddot{a}_{\overline{n-k+1}|}$, payable at time $t = k$ if death occurs in year k $(1 \leq k \leq n)$.

If the family income benefit is purchased by a single premium P, we have

$$P = a_{\overline{n}|} - a_{x:\overline{n}|}.$$

The prospective reserve is

$${}_tV^{prosp} = 1 + a_{\overline{n-t}|} - (1 + a_{x+t:\overline{n-t}|}) = a_{\overline{n-t}|} - a_{x+t:\overline{n-t}|}$$

and the retrospective reserve is

$${}_tV^{retro} = \frac{P - \sum\limits_{k=1}^{t} \dfrac{C_{x+k-1}}{D_x}\, \ddot{a}_{\overline{n-k+1}|}}{{}_tE_x}.$$

If the premium is paid annually in the form of an annuity-due, the annual premium P is

$$P = \frac{a_{\overline{n}|} - a_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}},$$

so

$${}_tV^{prosp} = a_{\overline{n-t}|} - a_{x+t:\overline{n-t}|} - P\,\ddot{a}_{x+t:\overline{n-t}|},$$

and

$${}_tV^{retro} = \frac{P\,\ddot{a}_{x:\overline{t}|} - \sum_{k=1}^{t} \frac{C_{x+k-1}}{D_x}\,\ddot{a}_{\overline{n-k+1}|}}{{}_tE_x}.$$

Finally, let us examine an n year term certain insurance of \$1 whose premium is payable annually. If the insured dies in year k $(1 \le k \le n)$, the present value of the benefit at $t = k$ is v^{n-k}. So, the insurance is equivalent to an n year insurance with a survival benefit of \$1 payable at the end of year n, and a death benefit of v^{n-k}, payable at time $t = k$, if death occurs in year k $(1 \le k \le n)$. Denoting the annual premium payable yearly in advance by P, we get

$$P = \frac{v^n}{\ddot{a}_{x:\overline{n}|}},$$

so

$${}_tV^{prosp} = v^{n-t} - P\,\ddot{a}_{x+t:\overline{n-t}|},$$

and

$${}_tV^{retro} = \frac{P\,\ddot{a}_{x:\overline{t}|} - \sum_{k=1}^{t} \frac{C_{x+k-1}}{D_x}\,v^{n-k}}{{}_tE_x}.$$

As we can see, the method we have just presented usually gives a rather complicated formula for the retrospective reserve. We can often simplify it in the following way. Instead of capitalizing the future benefits at the time of death or at the end of the year of death, we can allow the benefit payments to go on until time t and then capitalize the outstanding benefits.

Let us consider the n year annuity-immediate of \$1 per annum with guaranteed payments in the first m years.

If $t \le m$ then the present value at $t_0 = 0$ of the benefit payments before t is $a_{\overline{t-1}|}$. Note that the payment at time t does not belong here since it is a survival benefit. If the insured dies before t, there are still $m - t + 1$ annuity payments outstanding at time t whose present value at t is $\ddot{a}_{\overline{m-t+1}|}$. Considering this as a single death benefit payable at t, its expected present value at time $t_0 = 0$ is

$$\begin{aligned} {}_tq_x\, v^t\, \ddot{a}_{\overline{m-t+1}|} &= (v^t - v^t\, {}_tp_x)\, \ddot{a}_{\overline{m-t+1}|} \\ &= (v^t - {}_tE_x)\, \ddot{a}_{\overline{m-t+1}|}. \end{aligned}$$

Therefore, the retrospective reserve is

$${}_tV^{retro} = \frac{P - (a_{\overline{t-1}|} + (v^t - {}_tE_x)\, \ddot{a}_{\overline{m-t+1}|})}{{}_tE_x} \quad \text{if } t \le m.$$

If $t > m$, then the present value at $t_0 = 0$ of the benefit payments before t is $a_{\overline{m}|} + {}_{m|}a_{x:\overline{t-m-1}|}$. If the insured dies before t, there are no benefits payable after t. Thus,

$${}_tV^{retro} = \frac{P - (a_{\overline{m}|} + {}_{m|}a_{x:\overline{t-m-1}|})}{{}_tE_x}, \quad \text{if } t > m.$$

If we consider the n year family income benefit of $1 per annum, the present value at time $t_0 = 0$ of the benefit payments before t is $a_{\overline{t-1}|} - a_{x:\overline{t-1}|}$. If the insured dies before t, the present value of the outstanding payments after t is $\ddot{a}_{\overline{n-t+1}|}$. Regarding this as a one time death benefit payable at t, its expected present value at time $t_0 = 0$ is

$${}_tq_x\, v^t\, \ddot{a}_{\overline{n-t+1}|} = (v^t - {}_tE_x)\, \ddot{a}_{\overline{n-t+1}|}.$$

Thus, if a single premium is paid, the retrospective reserve can be expressed as

$${}_tV^{retro} = \frac{P - (a_{\overline{t-1}|} - a_{x:\overline{t-1}|} + (v^t - {}_tE_x)\, \ddot{a}_{\overline{n-t+1}|})}{{}_tE_x},$$

where P denotes the single premium.

If the premium is paid annually in the form of an annuity-due, then

$${}_tV^{retro} = \frac{P\, \ddot{a}_{x:\overline{t}|} - (a_{\overline{t-1}|} - a_{x:\overline{t-1}|} + (v^t - {}_tE_x)\, \ddot{a}_{\overline{n-t+1}|})}{{}_tE_x},$$

where P is the annual premium.

In the case of the n year term certain insurance of $1, whose premium is paid annually, there are not any benefit payments before t, if $t \le n$. On the other hand, if the insured dies before t, the present value of the outstanding

payments after t is v^{n-t}. Regarding this as a death benefit payable at t, its expected present value at time $t_0 = 0$ is

$$ {}_tq_x\, v^t\, v^{n-t} = (v^t - {}_tE_x)\, v^{n-t} = v^n - {}_tE_x\, v^{n-t}. $$

Thus the retrospective reserve at duration t is

$$ {}_tV^{retro} = \frac{P\,\ddot{a}_{x:\overline{t}|} - (v^n - {}_tE_x\, v^{n-t})}{{}_tE_x} $$

where P is the annual premium.

If the (premium and benefit) annuities are paid continuously at a constant rate, we get the following expressions for the reserves.

In the case of an n year continuous annuity payable at a rate of \$1 per annum, with guaranteed payments in the first m years, the prospective reserve is

$$ {}_tV^{prosp} = \bar{a}_{\overline{m-t}|} + {}_{m-t|}\bar{a}_{x+t:\overline{n-m}|},\ \text{if } t \le m $$

and

$$ {}_tV^{prosp} = \bar{a}_{x+t:\overline{n-t}|},\ \text{if } t > m. $$

Moreover, the retrospective reserve is

$$ {}_tV^{retro} = \frac{P - (\bar{a}_{\overline{t}|} + (v^t - {}_tE_x)\,\bar{a}_{\overline{m-t}|})}{{}_tE_x},\ \text{if } t \le m $$

and

$$ {}_tV^{retro} = \frac{P - (\bar{a}_{\overline{m}|} + {}_{m|}\bar{a}_{x:\overline{t-m}|})}{{}_tE_x},\ \text{if } t > m, $$

where P is the single premium:

$$ P = \bar{a}_{\overline{m}|} + {}_{m|}\bar{a}_{x:\overline{n-m}|}. $$

If we consider an n year family income benefit payable continuously at a rate of \$1 per annum, purchased by a single premium, the prospective reserve is

$$ {}_tV^{prosp} = \bar{a}_{\overline{n-t}|} - \bar{a}_{x+t:\overline{n-t}|} $$

and the retrospective reserve is

$$ {}_tV^{retro} = \frac{P - (\bar{a}_{\overline{t|}} - \bar{a}_{x:\overline{t|}} + (v^t - {}_tE_x)\,\bar{a}_{\overline{n-t|}})}{{}_tE_x}, $$

where P is the single premium

$$ P = \bar{a}_{\overline{n|}} - \bar{a}_{x:\overline{n|}}. $$

If the premium is paid continuously throughout the duration of the insurance then the prospective reserve is

$$ {}_tV^{prosp} = \bar{a}_{\overline{n-t|}} - \bar{a}_{x+t:\overline{n-t|}} - P\,\bar{a}_{x+t:\overline{n-t|}} $$

and the retrospective reserve is

$$ {}_tV^{retro} = \frac{P\,\bar{a}_{x:\overline{t|}} - (\bar{a}_{\overline{t|}} - \bar{a}_{x:\overline{t|}} + (v^t - {}_tE_x)\,\bar{a}_{\overline{n-t|}})}{{}_tE_x}, $$

where P is the annual premium:

$$ P = \frac{\bar{a}_{\overline{n|}} - \bar{a}_{x:\overline{n|}}}{\bar{a}_{x:\overline{n|}}}. $$

Considering an n year term certain insurance of \$1, whose premium is payable continuously at a constant rate, we find that the prospective reserve is

$$ {}_tV^{prosp} = v^{n-t} - P\,\bar{a}_{x+t:\overline{n-t|}} $$

and the retrospective reserve is

$$ {}_tV^{retro} = \frac{P\,\bar{a}_{x:\overline{t|}} - (v^n - {}_tE_x\, v^{n-t})}{{}_tE_x}, $$

where P is the annual premium:

$$ P = \frac{v^n}{\bar{a}_{x:\overline{n|}}}. $$

EXAMPLE 1.3. The first 5 payments of a 15 year annuity-immediate of \$800 per annum are guaranteed. The annuity is purchased by a single premium at the age of 35. Find the expressions for the prospective and retrospective reserves at the end of each policy year. Based on a 6% annual rate of interest, evaluate them numerically at the end of years 3 and 10.

Solution: The single premium P can be obtained as

$$\begin{aligned} P &= 800\,(a_{\overline{5}|} + {}_{5|}a_{35:\overline{10}|}) \\ &= 800\left(a_{\overline{5}|} + \frac{N_{41} - N_{51}}{D_{35}}\right) \\ &= 800\left(4.2124 + \frac{125101.93 - 59608.16}{12256.76}\right) \\ &= \$7644.71. \end{aligned}$$

The prospective reserve is

$$\begin{aligned} {}_tV^{prosp} &= 800\,(\ddot{a}_{\overline{6-t}|} + {}_{5-t|}a_{35+t:\overline{10}|}) \\ &= 800\left(1 + a_{\overline{5-t}|} + \frac{N_{41} - N_{51}}{D_{35+t}}\right) \quad \text{if } t \le 5 \end{aligned}$$

and

$${}_tV^{prosp} = 800\,(\ddot{a}_{35+t:\overline{16-t}|}) = 800\left(\frac{N_{35+t} - N_{51}}{D_{35+t}}\right) \quad \text{if } t > 5.$$

Furthermore, the retrospective reserve is

$$\begin{aligned} {}_tV^{retro} &= \frac{P - 800\,(a_{\overline{t-1}|} + (v^t - {}_tE_{35})\,\ddot{a}_{\overline{6-t}|})}{{}_tE_{35}} \\ &= \frac{D_{35}}{D_{35+t}}\left[P - 800\left(a_{\overline{t-1}|} + \left(v^t - \frac{D_{35+t}}{D_{35}}\right)\ddot{a}_{\overline{6-t}|}\right)\right] \quad \text{if } t \le 5 \end{aligned}$$

and

$$\begin{aligned} {}_tV^{retro} &= \frac{P - 800\,(a_{\overline{5}|} + {}_{5|}a_{35:\overline{t-6}|})}{{}_tE_{35}} \\ &= \frac{D_{35}}{D_{35+t}}\left(P - 800\left(a_{\overline{5}|} + \frac{N_{41} - N_{35+t}}{D_{35}}\right)\right), \quad \text{if } t > 5. \end{aligned}$$

At the end of year 3, the prospective reserve is

$$
\begin{aligned}
{}_3V^{prosp} &= 800\left(1 + a_{\overline{2}|} + \frac{N_{41} - N_{51}}{D_{38}}\right) \\
&= 800\left(1 + 1.8334 + \frac{125101.93 - 59608.16}{10224.96}\right) \\
&= \$7390.95,
\end{aligned}
$$

and the retrospective reserve is

$$
\begin{aligned}
&{}_3V^{retro} \\
&= \frac{D_{35}}{D_{38}}\left(P - 800\left(a_{\overline{2}|} + \left(v^3 - \frac{D_{38}}{D_{35}}\right)\ddot{a}_{\overline{3}|}\right)\right) \\
&= \frac{12256.76}{10224.96}\left[7644.71 - 800\left(1.8334 + \left(0.83962 - \frac{10224.96}{12256.76}\right)2.8334\right)\right] \\
&= \$7390.97.
\end{aligned}
$$

The $0.02 difference between the prospective and retrospective reserves is due to round-off errors.

At the end of year 10, the prospective reserve is

$$
\begin{aligned}
{}_{10}V^{prosp} &= 800\frac{N_{45} - N_{51}}{D_{45}} \\
&= 800\frac{93953.92 - 59608.16}{6657.69} \\
&= \$4127.05,
\end{aligned}
$$

and the retrospective reserve is

$$
\begin{aligned}
{}_{10}V^{retro} &= \frac{D_{35}}{D_{45}}\left(P - 800\left(a_{\overline{5}|} + \frac{N_{41} - N_{45}}{D_{35}}\right)\right) \\
&= \frac{12256.76}{6657.69}\left[7644.71 - 800\left(4.2124 + \frac{125101.93 - 93953.92}{12256.76}\right)\right] \\
&= \$4127.06.
\end{aligned}
$$

EXAMPLE 1.4. A family income benefit of $5000 per annum is issued to a life aged 30 for a term of 20 years. A single premium is paid at the time of the purchase. Find the expressions for the prospective and retrospective reserves at the end of each policy year. Evaluate them numerically at the end of year 8, based on a 6% annual interest rate.

Solution: The single premium is

$$P = 5000\ (a_{\overline{20}|} - a_{30:\overline{20}|})$$

$$= 5000\left(a_{\overline{20}|} - \frac{N_{31} - N_{51}}{D_{30}}\right)$$
$$= 5000\left(11.4699 - \frac{245762.85 - 59608.16}{16542.86}\right)$$
$$= \$1085.14.$$

The prospective reserve at the end of year t is

$$_tV^{prosp} = 5000(a_{\overline{20-t}|} - a_{30+t:\overline{20-t}|})$$
$$= 5000\left(a_{\overline{20-t}|} - \frac{N_{31+t} - N_{51}}{D_{30+t}}\right).$$

The retrospective reserve at duration t is

$$_tV^{retro} = \frac{P - 5000\,(a_{\overline{t-1}|} - a_{30:\overline{t-1}|} + (v^t - {}_tE_x)\,\ddot{a}_{\overline{21-t}|})}{_tE_{30}}$$
$$= \frac{D_{30}}{D_{30+t}}\left[P - 5000\left(a_{\overline{t-1}|} - \frac{N_{31} - N_{30+t}}{D_{30}} + \left(v^t - \frac{D_{30+t}}{D_{30}}\right)(1 + a_{\overline{20-t}|})\right)\right].$$

The prospective reserve at the end of year 8 is

$$_8V^{prosp} = 5000\left(a_{\overline{12}|} - \frac{N_{39} - N_{51}}{D_{38}}\right)$$
$$= 5000\left(8.3838 - \frac{143779.13 - 59608.16}{10224.96}\right)$$
$$= \$759.44,$$

and the retrospective reserve is

$$_8V^{retro} = \frac{D_{30}}{D_{38}}\left[1085.14 - 5000\left(a_{\overline{7}|} - \frac{N_{31} - N_{38}}{D_{30}} + \left(v^8 - \frac{D_{38}}{D_{30}}\right)(1 + a_{\overline{12}|})\right)\right]$$
$$= \frac{16542.86}{10224.96}\left[1085.14 - 5000\left(5.5824 - \frac{245762.85 - 154004.08}{16542.86}\right.\right.$$
$$\left.\left. + \left(0.62741 - \frac{10224.96}{16542.86}\right)9.3838\right)\right]$$
$$= \$759.53.$$

EXAMPLE 1.5. The premium for a 10 year term certain insurance of \$9000 issued to a life aged 50 is payable continuously. Find the expressions for the prospective and retrospective reserves at the end of each policy year.

Using a 6% annual rate of interest, evaluate them numerically at the end of year 8.

Solution: The annual premium is

$$P = 9000 \frac{v^{10}}{\bar{a}_{50:\overline{10}|}}.$$

Now,

$$v^{10} = 0.55839$$

and

$$\begin{aligned}\bar{a}_{50:\overline{10}|} &\approx \ddot{a}_{50:\overline{10}|} - \frac{1}{2}(1 - {}_{10}E_{50})\\ &= \frac{N_{50} - N_{60}}{D_{50}} - \frac{1}{2}\left(1 - \frac{D_{60}}{D_{50}}\right)\\ &= \frac{64467.45 - 27664.55}{4859.30} - \frac{1}{2}\left(1 - \frac{2482.16}{4859.30}\right)\\ &= 7.32911\,.\end{aligned}$$

Thus,

$$P = 9000\,\frac{0.55839}{7.32911} = \$685.69.$$

The prospective reserve at duration t is

$${}_tV^{prosp} = 9000\, v^{10-t} - P\, \bar{a}_{50+t:\overline{10-t}|}$$

and the retrospective reserve is

$${}_tV^{retro} = \frac{P\, \bar{a}_{50:\overline{t}|} - 9000\,(v^{10} - {}_tE_x\, v^{10-t})}{{}_tE_x}.$$

At the end of year 8, the prospective reserve is

$${}_8V^{prosp} = 9000\, v^2 - P\, \bar{a}_{58:\overline{2}|},$$

where

$$v^2 = 0.89000$$

and

$$
\begin{aligned}
\bar{a}_{58:\overline{2}|} &\approx \ddot{a}_{58:\overline{2}|} - \frac{1}{2}(1 - {}_2E_{58}) \\
&= \frac{N_{58} - N_{60}}{D_{58}} - \frac{1}{2}\left(1 - \frac{D_{60}}{D_{58}}\right) \\
&= \frac{33186.93 - 27664.55}{2857.67} - \frac{1}{2}\left(1 - \frac{2482.16}{2857.67}\right) \\
&= 1.86677.
\end{aligned}
$$

Therefore,

$$
{}_8V^{prosp} = 9000 \times 0.89000 - 685.59 \times 1.86677 = \$6730.16.
$$

The retrospective reserve at the end of year 8 is

$$
{}_8V^{retro} = \frac{P\,\bar{a}_{50:\overline{8}|} - 9000\,(v^{10} - {}_8E_{50}\,v^2)}{{}_8E_{50}}.
$$

Here, we have

$$v^2 = 0.89000,$$

$$v^{10} = 0.55839,$$

$$
{}_8E_{50} = \frac{D_{58}}{D_{50}} = \frac{2857.67}{4859.30} = 0.5880826,
$$

and

$$
\begin{aligned}
\bar{a}_{50:\overline{8}|} &\approx \ddot{a}_{50:\overline{8}|} - \frac{1}{2}(1 - {}_8E_{50}) \\
&= \frac{N_{50} - N_{58}}{D_{50}} - \frac{1}{2}\left(1 - \frac{D_{58}}{D_{50}}\right) \\
&= \frac{64467.45 - 33186.93}{4859.30} - \frac{1}{2}\left(1 - \frac{2857.67}{4859.30}\right) \\
&= 6.23129.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
{}_8V^{retro} &= \frac{685.59 \times 6.23129 - 9000\,(0.55839 - 0.5880826 \times 0.89000)}{0.5880826} \\
&= \$6728.88.
\end{aligned}
$$

Recall that in Section 4.1, we introduced some special notations for annual premiums. There are corresponding symbols for the reserves, as well. They can be obtained by replacing P by ${}_tV$ in the respective expression. If there was already a prefix attached to P, it is raised above t. For example, taking an n-year endowment, its annual premium is denoted by $P_{x:\overline{n|}}$ or $P(A_{x:\overline{n|}})$, so the reserve at time t is ${}_tV_{x:\overline{n|}}$ or ${}_tV(A_{x:\overline{n|}})$. As another example, consider an n year term insurance whose premium is payable for m years only. Then the symbol for the annual premium is ${}_mP^{1}_{x:\overline{n|}}$ or ${}_mP(A^{1}_{x:\overline{n|}})$ thus the reserve at time t is denoted by ${}^{m}_{t}V^{1}_{x:\overline{n|}}$ or ${}^{m}_{t}V(A^{1}_{x:\overline{n|}})$. Using the formulas for the annual premiums given in Section 4.1 it is easy to find expressions for the reserves. Let us see some examples. In all of the examples, we assume that t is an integer. We will use the prospective method to determine the reserves. The reader is encouraged to derive the formulas for the retrospective reserves.

In the case of a pure endowment, we have

$$
\begin{aligned}
{}_tV_{x:\overline{n|}}^{\;\;1} &= {}_tV\,(A_{x:\overline{n|}}^{\;\;1}) \\
&= A_{x+t:\overline{n-t|}}^{\;\;1} - P_{x:\overline{n|}}^{\;\;1}\,\ddot{a}_{x+t:\overline{n-t|}} \\
&= A_{x+t:\overline{n-t|}}^{\;\;1} - A_{x:\overline{n|}}^{\;\;1}\,\frac{\ddot{a}_{x+t:\overline{n-t|}}}{\ddot{a}_{x:\overline{n|}}}\,.
\end{aligned}
$$

For an n-year term insurance we get

$$
\begin{aligned}
{}_tV^{1}_{x:\overline{n|}} &= {}_tV(A^{1}_{x:\overline{n|}}) \\
&= A^{1}_{x+t:\overline{n-t|}} - P^{1}_{x:\overline{n|}}\,\ddot{a}_{x+t:\overline{n-t|}} \\
&= A^{1}_{x+t:\overline{n-t|}} - A^{1}_{x:\overline{n|}}\,\frac{\ddot{a}_{x+t:\overline{n-t|}}}{\ddot{a}_{x:\overline{n|}}}\,,
\end{aligned}
$$

and if n is infinity, we get

$$
\begin{aligned}
{}_tV_x &= {}_tV(A_x) \\
&= A_{x+t} - P_x\,\ddot{a}_{x+t} \\
&= A_{x+t} - A_x\,\frac{\ddot{a}_{x+t}}{\ddot{a}_x} \\
&= 1 - d\,\ddot{a}_{x+t} - (1 - d\,\ddot{a}_x)\,\frac{\ddot{a}_{x+t}}{\ddot{a}_x} \\
&= 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}\,.
\end{aligned}
$$

If we consider an n-year endowment insurance, with the death benefit payable at the end of the year of death, we obtain

$$
\begin{aligned}
{}_tV_{x:\overline{n}|} &= {}_tV\,(A_{x:\overline{n}|}) \\
&= A_{x+t:\overline{n-t}|} - P_{x:\overline{n}|}\,\ddot{a}_{x+t:\overline{n-t}|} \\
&= A_{x+t:\overline{n-t}|} - A_{x:\overline{n}|}\,\frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}} \\
&= 1 - d\,\ddot{a}_{x+t:\overline{n-t}|} - (1 - d\,\ddot{a}_{x:\overline{n}|})\,\frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}} \\
&= 1 - \frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}} \,.
\end{aligned}
$$

If the premium is paid continuously, we can also obtain some nice expressions. For example, in the case of an n year endowment insurance whose death benefit is payable at the moment of death, we have

$$
\begin{aligned}
{}_t\bar{V}\,(\bar{A}_{x:\overline{n}|}) &= \bar{A}_{x+t:\overline{n-t}|} - \bar{P}\,(\bar{A}_{x:\overline{n}|})\,\bar{a}_{x+t:\overline{n-t}|} \\
&= \bar{A}_{x+t:\overline{n-t}|} - \bar{A}_{x:\overline{n}|}\,\frac{\bar{a}_{x+t:\overline{n-t}|}}{\bar{a}_{x:\overline{n}|}} \\
&= (1 - \delta\,\bar{a}_{x+t:\overline{n-t}|}) - (1 - \delta\,\bar{a}_{x:\overline{n}|})\,\frac{\bar{a}_{x+t:\overline{n-t}|}}{\bar{a}_{x:\overline{n}|}} \\
&= 1 - \frac{\bar{a}_{x+t:\overline{n-t}|}}{\bar{a}_{x:\overline{n}|}} \,,
\end{aligned}
$$

and if n is infinity, we get

$$
{}_t\bar{V}\,(\bar{A}_x) = 1 - \frac{\bar{a}_{x+t}}{\bar{a}_x} \,.
$$

Using the results of Section 4.1, we can derive some more general formulas for the reserves. Let EPV be the present value of the benefits of an insurance at the start of the insurance. Moreover, assume that the premium is paid in the form of an m-year annuity. Let us select a duration t, which falls into the term of the insurance. Let EPV_1 denote the expected present value, at the start of the insurance, of the benefits that are payable before t, and EPV_2 denote the expected present value at t of the benefits that are payable after t. The following formulas assume that payments after t cannot be made unless the insured survives to t. Otherwise, the formulas have to be adjusted using the techniques described earlier in this section. Then, we obtain the following results.

If the premiums are payable annually in advance, then

$$
{}_mP(EPV) = \frac{EPV}{\ddot{a}_{x:\overline{m}|}} \,.
$$

Thus,

$$
{}^{m}_{t}V^{prosp}(EPV) = \begin{cases} EPV_2 - {}_mP(EPV)\ddot{a}_{x+t:\overline{m-t}|} = EPV_2 - EPV\dfrac{\ddot{a}_{x+t:\overline{m-t}|}}{\ddot{a}_{x:\overline{m}|}}, & \text{if } t \le m \\ EPV_2, \text{ if } t > m. \end{cases}
$$

Moreover,

$$
{}^{m}_{t}V^{retro}(EPV) = \begin{cases} \dfrac{1}{{}_tE_x}(\ddot{a}_{x:\overline{t}|}\,{}_mP(PEV) - EPV_1) = \dfrac{1}{{}_tE_x}\left(EPV\dfrac{\ddot{a}_{x:\overline{t}|}}{\ddot{a}_{x:\overline{m}|}} - EPV_1\right), & \text{if } t \le m \\ \dfrac{1}{{}_tE_x}(EPV - EPV_1), \text{ if } t > m. \end{cases}
$$

If the premiums are payable $p$thly in advance, then

$$
{}_mP^{(p)}(EPV) = \frac{EPV}{\ddot{a}^{(p)}_{x:\overline{m}|}}.
$$

So,

$$
{}^{m}_{t}V^{(p)prosp}(EPV) = \begin{cases} EPV_2 - {}_mP^{(p)}(EPV)\,\ddot{a}^{(p)}_{x+t:\overline{m-t}|} = EPV_2 - EPV\dfrac{\ddot{a}^{(p)}_{x+t:\overline{m-t}|}}{\ddot{a}^{(p)}_{x:\overline{m}|}}, & \text{if } t \le m \\ EPV_2, \text{ if } t > m. \end{cases}
$$

Furthermore,

$$
{}^{m}_{t}V^{(p)retro}(EPV) = \begin{cases} \dfrac{1}{{}_tE_x}(\ddot{a}^{(p)}_{x:\overline{t}|}\,{}_mP^{(p)}(EPV) - EPV_1) = \dfrac{1}{{}_tE_x}\left(EPV\dfrac{\ddot{a}^{(p)}_{x:\overline{t}|}}{\ddot{a}^{(p)}_{x:\overline{m}|}} - EPV_1\right), & \text{if } t \le m \\ \dfrac{1}{{}_tE_x}(EPV - EPV_1), \text{ if } t > m. \end{cases}
$$

If the premiums are payable continuously, then

$$_m\overline{P}\,(EPV) = \frac{EPV}{\overline{a}_{x:\overline{m}|}}.$$

Hence,

$$\begin{aligned}&{}^m_t\overline{V}^{prosp}(EPV)\\ &= \begin{cases} EPV_2 - {}_m\overline{P}\,(EPV)\,\overline{a}_{x+t:\overline{m-t}|} = EPV_2 - EPV\dfrac{\overline{a}_{x+t:\overline{m-t}|}}{\overline{a}_{x:\overline{m}|}}, & \text{if } t \le m\\ EPV_2, \text{ if } t > m. \end{cases}\end{aligned}$$

Moreover,

$$\begin{aligned}&{}^m_t\overline{V}^{retro}(EPV)\\ &= \begin{cases} \dfrac{1}{_tE_x}\left(\overline{a}_{x:\overline{t}|}\,\overline{P}\,(EPV) - EPV_1\right) = \dfrac{1}{_tE_x}\left(EPV\dfrac{\overline{a}_{x:\overline{t}|}}{\overline{a}_{x:\overline{m}|}} - EPV_1\right), & \text{if } t \le m\\ \dfrac{1}{_tE_x}(EPV - EPV_2), \text{ if } t > m. \end{cases}\end{aligned}$$

Note that in all these cases, for any fixed t, the reserve ${}^m_tV$ is a decreasing function in m. We will only prove this when the premium is paid yearly in advance, but the proof works in exactly the same way for the other two cases as well.

So, let us pick m and m' such that $m < m'$. We have to prove that

$${}^m_tV \ge {}^{m'}_tV.$$

If $t \le m$, we get

$${}^m_tV^{retro}(EPV) = \frac{1}{_tE_x}\left(EPV\frac{\ddot{a}_{x:\overline{t}|}}{\ddot{a}_{x:\overline{m}|}} - EPV_1\right)$$

and

$${}^{m'}_tV^{retro}(EPV) = \frac{1}{_tE_x}\left(EPV\frac{\ddot{a}_{x:\overline{t}|}}{\ddot{a}_{x:\overline{m'}|}} - EPV_1\right).$$

Now, $m \le m'$ implies

$$\ddot{a}_{x:\overline{m}|} \le \ddot{a}_{x:\overline{m'}|},$$

thus

$$\frac{1}{\ddot{a}_{x:\overline{m}|}} \ge \frac{1}{\ddot{a}_{x:\overline{m'}|}}.$$

Therefore,

$${}^{m}_{t}V^{retro}(EPV) \ge {}^{m'}_{t}V^{retro}(EPV).$$

If $m < t \le m'$, we have

$${}^{m}_{t}V^{retro}(EPV) = \frac{1}{{}_tE_x}(EPV - EPV_1)$$

and

$${}^{m'}_{t}V^{retro}(EPV) = \frac{1}{{}_tE_x}\left(EPV\frac{\ddot{a}_{x:\overline{t}|}}{\ddot{a}_{x:\overline{m'}|}} - EPV_1\right).$$

Since $t \le m'$, we get

$$\ddot{a}_{x:\overline{t}|} \le \ddot{a}_{x:\overline{m'}|},$$

hence

$$1 \ge \frac{\ddot{a}_{x:\overline{t}|}}{\ddot{a}_{x:\overline{m'}|}}.$$

Thus,

$${}^{m}_{t}V^{retro}(EPV) \ge {}^{m'}_{t}V^{retro}(EPV).$$

If $m' < t$, we obtain

$${}^{m}_{t}V^{retro}(EPV) = \frac{1}{{}_tE_x}(EPV - EPV_1)$$

and

$$ {}^{m'}_{t}V^{retro}(EPV) = \frac{1}{{}_tE_x}(EPV - EPV_1). $$

Therefore,

$$ {}^{m'}_{t}V^{retro}(EPV) = {}^{m}_{t}V^{retro}(EPV), $$

and

$$ {}^{m'}_{t}V^{retro}(EPV) \geq {}^{m}_{t}V^{retro}(EPV) $$

is satisfied again.

In other words, we can say that if we shorten the premium payment period, the reserve at any duration will increase or remain unchanged.

So far, we have studied insurances where the prospective method was easier to use than the retrospective. However, there are situations where the retrospective method is simpler. For example, consider an n year annuity-due deferred for ℓ years, whose premium is payable for m years ($m \leq \ell$). If we want to find the reserve at the end of any of the first m years; that is, $t \leq m$, then we have

$$ \begin{aligned} {}^{m}_{t}V^{prosp}({}_{\ell|}\ddot{a}_{x:\overline{n}|}) &= {}_{\ell-t|}\ddot{a}_{x+t:\overline{n}|} - {}_mP({}_{\ell|}\ddot{a}_{x:\overline{n}|})\, \ddot{a}_{x+t:\overline{m-t}|} \\ &= {}_{\ell-t|}\ddot{a}_{x+t:\overline{n}|} - {}_{\ell|}\ddot{a}_{x:\overline{n}|}\frac{\ddot{a}_{x+t:\overline{m-t}|}}{\ddot{a}_{x:\overline{m}|}} \end{aligned} $$

and

$$ {}^{m}_{t}V^{retro}({}_{\ell|}\ddot{a}_{x:\overline{n}|}) = \frac{1}{{}_tE_x}\, {}_mP({}_{\ell|}\ddot{a}_{x:\overline{n}|})\, \ddot{a}_{x:\overline{t}|} = \frac{D_x}{D_{x+t}}\, {}_{\ell|}\ddot{a}_{x:\overline{n}|}\, \frac{\ddot{a}_{x:\overline{t}|}}{\ddot{a}_{x:\overline{m}|}}, $$

so the retrospective reserve has a simpler form. This is not surprising since no benefit payments can take place before t, so in the retrospective reserve we only have to take into account the premium payments. If we want to determine the reserve after the premium payment period, but before the annuity payment period; that is at a t for which $m < t \leq \ell$, we can use both methods quite easily. They give

$$ {}^{m}_{t}V^{prosp}({}_{\ell|}\ddot{a}_{x:\overline{n}|}) = {}_{\ell-t|}\ddot{a}_{x+t:\overline{n}|} $$

and

$$ {}_t^m V^{retro}(\ell \mid \ddot{a}_{x:\overline{n}|}) = \frac{1}{{}_tE_x}\, {}_mP(\ell \mid \ddot{a}_{x:\overline{n}|})\, \ddot{a}_{x:\overline{m}|} $$
$$ = \frac{D_x}{D_{x+t}}\, \ell \mid \ddot{a}_{x:\overline{n}|}. $$

If we are interested in the reserve after the annuity payments have started; that is, $t > \ell$, the prospective method is easier to apply, since there are not any premium payments in the future to be concerned about. We have

$$ {}_t^m V^{prosp}(\ell \mid \ddot{a}_{x:\overline{n}|}) = \ddot{a}_{x+t:\overline{n-t}|} $$

and

$$ {}_t^m V^{retro}(\ell \mid \ddot{a}_{x:\overline{n}|}) = \frac{1}{{}_tE_x}\left({}_mP(\ell \mid \ddot{a}_{x:\overline{n}|})\, \ddot{a}_{x:\overline{m}|} - \ell \mid \ddot{a}_{x:\overline{t-\ell}|}\right) $$
$$ = \frac{D_x}{D_{x+t}}\left(\ell \mid \ddot{a}_{x:\overline{n}|} - \ell \mid \ddot{a}_{x:\overline{t-\ell}|}\right). $$

EXAMPLE 1.6. A 20 year term insurance of \$2000 is issued to a life aged 45. The level premium is payable annually. What is the reserve at the end of year 12? Use a 6% annual rate of interest.

Solution: The reserve is $2000\ {}_{12}V^{1}_{45:\overline{20}|}$, where

$$ {}_{12}V^{1}_{45:\overline{20}|} = A^{1}_{57:\overline{8}|} - A^{1}_{45:\overline{20}|}\ \frac{\ddot{a}_{57:\overline{8}|}}{\ddot{a}_{45:\overline{20}|}}. $$

Now,

$$ A^{1}_{57:\overline{8}|} = \frac{M_{57} - M_{65}}{D_{57}} = \frac{1009.8515 - 750.5749}{3061.66} = 0.0846850, $$

$$ A^{1}_{45:\overline{20}|} = \frac{M_{45} - M_{65}}{D_{45}} = \frac{1339.5427 - 750.5749}{6657.69} = 0.0884643, $$

$$ \ddot{a}_{57:\overline{8}|} = \frac{N_{57} - N_{65}}{D_{57}} = \frac{36248.59 - 16890.50}{3061.66} = 6.32274, $$

and

$$ \ddot{a}_{45:\overline{20}|} = \frac{N_{45} - N_{65}}{D_{45}} = \frac{93953.92 - 16890.50}{6657.69} = 11.57510. $$

As a result,

$$_{12}V^{1}_{45:\overline{20}|} = 0.0846850 - 0.0884643\,\frac{6.32274}{11.57510} = 0.0363626.$$

So, the reserve at the end of year 12 is $2000 \times 0.0363626 = \$72.73$.

EXAMPLE 1.7. A whole life insurance of \$7000 is purchased for a life aged 50 by level annual premiums. Based on a 6% annual interest rate, calculate the reserve at the age of 63.

Solution: The reserve is $7000\,{}_{13}V_{50}$, where

$$_{13}V_{50} = 1 - \frac{\ddot{a}_{63}}{\ddot{a}_{50}}.$$

Now,

$$\ddot{a}_{50} = 13.26683$$

and

$$\ddot{a}_{63} = 10.40837,$$

so

$$_{13}V_{50} = 1 - \frac{10.40837}{13.26683} = 0.2154592.$$

Therefore, the reserve at the age of 63 is $7000 \times 0.2154592 = \$1508.21$.

EXAMPLE 1.8. A 20 year endowment of \$2000, with a death benefit payable at the moment of death, is issued to a life aged 35. The level premium is payable monthly for a term of 15 years. Find the reserves at the end of year 5 and year 17. Use a 6% annual rate of interest.

Solution: The annual premium is $2000\,{}_{15}P^{(12)}(\overline{A}_{35:\overline{20}|})$, where

$$_{15}P^{(12)}(\overline{A}_{35:\overline{20}|}) = \frac{\overline{A}_{35:\overline{20}|}}{\ddot{a}^{(12)}_{35:\overline{15}|}}.$$

Now,

$$\overline{A}_{35:\overline{20}|} = \frac{(1.06)^{\frac{1}{2}}(M_{35} - M_{55}) + D_{55}}{D_{35}}$$

$$= \frac{1.029563(1577.6833 - 1069.6405) + 3505.37}{12256.76}$$
$$= 0.3286702$$

and

$$\ddot{a}^{(12)}_{35:\overline{15}|} = \ddot{a}_{35:\overline{15}|} - \frac{12-1}{2 \times 12}(1 - {}_{15}E_{35})$$
$$= \frac{N_{35} - N_{50}}{D_{35}} - \frac{11}{24}\left(1 - \frac{D_{50}}{D_{35}}\right)$$
$$= \frac{188663.76 - 64467.45}{12256.76} - \frac{11}{24}\left(1 - \frac{4859.30}{12256.76}\right)$$
$$= 9.85626,$$

so

$$ {}_{15}P^{(12)}(\overline{A}_{35:\overline{20}|}) = \frac{0.3286702}{9.85626}$$
$$= 0.0333463.$$

Therefore, the monthly premium is $2000 \times 0.0333463 = \$66.69$.

At the end of year 5, the prospective reserve is

$$2000\left(\overline{A}_{40:\overline{15}|} - {}_{15}P^{(12)}(\overline{A}_{35:\overline{20}|})\,\ddot{a}^{(12)}_{40:\overline{10}|}\right).$$

Now,

$$\overline{A}_{40:\overline{15}|} = \frac{(1.06)^{\frac{1}{2}}(M_{40} - M_{55}) + D_{55}}{D_{40}}$$
$$= \frac{1.029563(1460.7038 - 1069.6405) + 3505.37}{9054.46}$$
$$= 0.4316099$$

and

$$\ddot{a}^{(12)}_{40:\overline{10}|} = \ddot{a}_{40:\overline{10}|} - \frac{12-1}{2 \times 12}(1 - {}_{10}E_{40})$$
$$= \frac{N_{40} - N_{50}}{D_{40}} - \frac{11}{24}\left(1 - \frac{D_{50}}{D_{40}}\right)$$
$$= \frac{134156.39 - 64467.45}{9054.46} - \frac{11}{24}\left(1 - \frac{4859.30}{9054.46}\right)$$

$$= 7.48428.$$

Thus, the reserve at the end of year 5 is

$$2000(0.4316099 - 0.0333463 \times 7.48428) = \$364.07.$$

The reserve at the end of year 17 is $2000\,\overline{A}_{52:\overline{3}|}$. The premiums do not play a role in this prospective reserve calculation since their payment stops in year 15. Now,

$$\begin{aligned}\overline{A}_{52:\overline{3}|} &= \frac{(1.06)^{\frac{1}{2}}(M_{52} - M_{55}) + D_{55}}{D_{52}} \\ &= \frac{1.029563(1155.4478 - 1069.6405) + 3505.37}{4271.55} \\ &= 0.8413138.\end{aligned}$$

Thus the reserve at the end of year 17 is

$$2000 \times 0.8413138 = \$1682.63.$$

EXAMPLE 1.9. A 15 year annuity-due of \$3000 per annum deferred 10 years is purchased for a life aged 55. The level annual premium is payable in the 10 years preceding the annuity payment period. Find the reserves at the end of year 5 and year 20 based on a 6% annual rate of interest.

Solution: The annual premium is $3000\ {}_{10|}\ddot{a}_{55:\overline{15}|}$. Now,

$${}_{10|}\ddot{a}_{55:\overline{15}|} = \frac{N_{65} - N_{80}}{D_{55}} = \frac{16890.50 - 2184.81}{3505.37} = 4.19519,$$

and hence the annual premium is $3000 \times 4.19519 = \$12585.57$.

At the end of year 5, the retrospective reserve is just the accumulation of the premium payments:

$$\begin{aligned}{}_5V^{retro} &= \frac{1}{{}_5E_{55}}\, 12585.57\, \ddot{a}_{55:\overline{5}|} \\ &= \frac{D_{55}}{D_{60}}\, 12585.57 \frac{N_{55} - N_{60}}{D_{55}} \\ &= 12585.57 \frac{N_{55} - N_{60}}{D_{60}} \\ &= 12585.57 \frac{43031.29 - 27664.55}{2482.16} \\ &= 77915.68.\end{aligned}$$

Therefore, the reserve at the end of year 5 is \$77915.68. At the end of year 20, using the prospective reserve is more convenient since the premium payments do not have to be taken into account.

$$
\begin{aligned}
{}_{20}V^{prosp} &= 3000\,\ddot{a}_{75:\overline{5}|} = 3000\frac{N_{75} - N_{80}}{D_{75}} = 3000\frac{4926.02 - 2184.81}{682.56} \\
&= 12048.22.
\end{aligned}
$$

Thus, the reserve at the end of year 20 is \$12048.22.

In the examples discussed so far, the reserves always turned out to be positive. However, in the case of certain insurances, we may get negative reserves as well. Next, we give some examples of negative reserves, point out the dangers involved in them, and show how to get rid of them.

First, consider a simple example. Assume the premium for a 1 year term insurance of \$1000 is payable at the end of the year. Then, the premium P can be determined from the equation

$$P\,A_{x:\overline{1}|}^{\;\;1} = 1000\,A_{x:\overline{1}|}^{1}.$$

Thus, $P = 1000\dfrac{A_{x:\overline{1}|}^{1}}{A_{x:\overline{1}|}^{\;\;1}}$.

Everything seems to work well with the premium. However, this is a potentially dangerous situation for the insurance company. The policyholder who survives to the end of the year will not receive any benefits, so he/she may decide to lapse the policy and not pay the premium. On the other hand, if the insured has died, the insurance company has to pay the death benefit. Therefore, the company may end up paying all the death benefits but not receiving any premiums. This problem can also be approached by examining the reserves. The (prospective) reserve at the end of the year will be $-P$ for any surviving policyholder. That means the expected present value of the future benefits is less than the expected present value of the future premiums. In other words, the insured can expect to receive less than he/she will pay. Therefore, lapsing the policy is in his/her own interest. On the other hand, the retrospective approach shows that the accumulation of the premium payments is less than the accumulation of the benefit payments. That means, on the average, the insurance company received less than it paid. So, it needs to receive further premiums in order to make up for this past deficiency. Thus, if the insured lapses the policy, the company will suffer a financial loss.

At the first glance, it may sound strange that although the company has not paid out any money to the withdrawing policyholders, their withdrawal means a financial loss to the company. However, we have to

keep in mind that pricing an insurance is based on the assumption that the actual death experience of the group of policyholders will follow a certain mortality table. If some policyholders withdraw, this leaves the company with a modified group of people whose mortality experience is worse than what can be expected from the mortality table. In other words, they form a biased sample instead of a representative sample of the population.

Remember that when we started discussing the reserves, we said that if we determine the reserve at a time t, when benefit and premium payments are possible, we calculate the reserve after the death benefits have been paid, but before the survival benefits and the premiums are paid. Now, we can see why we defined the reserve in this way. If a policyholder dies, the death benefit must be paid, so it has to be included among the payments of the past, which have already taken place by time t. On the other hand, if a policy is lapsed, the premium will not be paid and the company does not have to pay the survival benefit, either. Therefore, these payments cannot be treated as certain, so they have to be assigned to the future cash flow.

Note that if the insured withdraws the policy when the reserve is positive, the insurance company does not suffer a loss. In fact, the company may even return a certain part of the reserve to the policyholder.

Now, we give some very broad conditions under which the reserves cannot be negative.

If a single premium is paid at the commencement of the policy, the (prospective) reserve is nonnegative at any duration $t > 0$, since then the future cash flow only contains benefit payments and no premium payments and the present value of a nonnegative cash flow is nonnegative.

Next, assume the premiums for the insurance are payable over a certain time period. Assume that there exists a number m (not necessarily an integer), such that the premiums are payable before time $t = m$ and the benefits are payable after $t = m$. Note that this condition allows a survival benefit payment, but no death benefit or premium payment at time $t = m$. Then, the retrospective reserve is positive at any duration between 0 and m, since the past cash flow only contains premiums and the prospective reserve is positive at any duration greater than m because the future cash flow consists of benefits only. This is why ${}_tV_{\overset{1}{x}:\overline{n}|}$ and ${}^{k}_{t}V(\ell \mid \ddot{a}_{x:\overline{n}|})$ $(k \le \ell)$ are always nonnegative.

It is important to always keep in mind that, in our discussions, we study the reserve at time t, where t is a nonnegative integer with the property that payments after t cannot be made unless the insured has survived to time t. Otherwise, the previous statement would not be true. For example, consider the following insurance. The insurance is issued to a life aged x. If death occurs in the first year, a death benefit is payable at the end of the second year. The premium is payable at the end of the first year. We can see immediately that although the premium payments take place before $t = 1.5$ and the benefit payments after $t = 1.5$, the insurance is disadvantageous to the insurance company. Indeed, the policyholders who

survive to the end of the first year, will cancel their policies, since they will not receive any benefit payments but they are supposed to pay a premium. So, the company may have to pay all the death benefits without receiving any premiums. As a result, we are faced with a case of negative reserve. This can be seen easily if we use the technique introduced earlier for insurances whose payments can extend beyond the end of the year of death. What we have to do is to replace the death benefit payment at the end of the second year by its discounted value payable at the end of the first year. Then, the death benefit payment will precede the premium payment, so the negativity of the reserve at the end of the year 1 becomes obvious.

If the payments of the premiums and of the benefits are not separated by a fixed point in time, the insurance requires a more thorough investigation.

First, let us consider a whole life insurance with a constant death benefit and annual premium payments. Note that the probability of dying within one year increases as the insured grows older except for very young ages. For example, according to the mortality table in Appendix 2, q_x is monotone increasing in x for $x \geq 10$. Therefore, as the insured grows older, he/she is more likely to die and less likely to be alive. This implies that the insurance company pays out more and more in death benefits each year and receives less and less in premiums. Therefore, the company must have a positive reserve at any time from which the widening gap between future benefits and future premiums can be closed. We can also feel that the ratio of the reserve to the expected present value of future payments is increasing with time. Let us see how we can prove these statements in a mathematically correct way. We will consider a general situation that includes the whole life insurance as a special case.

THEOREM 1.1. *Consider an insurance issued to a life aged x at time zero. Assume that for any integer k, it is true that (premium and benefit) payments after time $t = k$ cannot be made, unless the insured survives to $t = k$. Let n be a positive integer or infinity, such that the probability of a premium payment at or after $t = n$ is zero. For any nonnegative integer $k \leq n - 1$, let us denote the expected present value at $t = k$ of the cash flow of premiums between times $t = k$ and $t = k + 1$ by P_k, and the expected present value at $t = k$ of the cash flow of benefits, between times $t = k$ and $t = k + 1$, by U_k. Moreover, if n is a finite number, U_n will denote the expected present value at $t = n$ of the cash flow of benefits after $t = n$. We will give P_k and U_k positive signs. Assume*

$$P_0 \geq P_1 \geq P_2 \geq \ldots \geq P_{n-1} > 0 \tag{8}$$

and

$$U_0 \leq U_1 \leq U_2 \leq \ldots \leq U_{n-1}. \tag{9}$$

Then, the function

$$r(t) = \frac{{}_tV}{\sum_{k=0}^{n-t-1} {}_kE_{x+t}\, P_{t+k}} \tag{10}$$

defined for $t = 0,1,\ldots,n-1,$ *is increasing in* t.

Proof: Using the definition of the prospective reserve, we get

$${}_tV = \sum_{k=0}^{n-t} {}_kE_{x+t}\, U_{t+k} - \sum_{k=0}^{n-t-1} {}_kE_{x+t}\, P_{t+k}, \text{ for } t = 0,1,\ldots,n-1. \tag{11}$$

Thus,

$$r(t) = \frac{\sum_{k=0}^{n-t} {}_kE_{x+t}\, U_{t+k} - \sum_{k=0}^{n-t-1} {}_kE_{x+t}\, P_{t+k}}{\sum_{k=0}^{n-t-1} {}_kE_{x+t}\, P_{t+k}}$$

$$= \frac{\sum_{k=0}^{n-t} {}_kE_{x+t}\, U_{t+k}}{\sum_{k=0}^{n-t-1} {}_kE_{x+t}\, P_{t+k}} - 1. \tag{12}$$

We have to prove

$$r(t) \le r(t+1), \text{ for } t = 0,1,\ldots,n-2. \tag{13}$$

In view of (12), inequality (13) is equivalent to

$$\frac{\sum_{k=0}^{n-t} {}_kE_{x+t}\, U_{t+k}}{\sum_{k=0}^{n-t-1} {}_kE_{x+t}\, P_{t+k}} \le \frac{\sum_{j=0}^{n-t-1} {}_jE_{x+t+1}\, U_{t+1+j}}{\sum_{j=0}^{n-t-2} {}_jE_{x+t+1}\, P_{t+1+j}}. \tag{14}$$

Since,

$${}_kE_{x+t} = {}_{k-1}E_{x+t+1}\ {}_1E_{x+t} \text{ if } k \ge 1,$$

(14) can be rewritten as

$$\frac{U_t + {}_1E_{x+t} \sum_{j=0}^{n-t-1} {}_jE_{x+t+1}\, U_{t+1+j}}{P_t + {}_1E_{x+t} \sum_{j=0}^{n-t-2} {}_jE_{x+t+1}\, P_{t+1+j}} \le \frac{\sum_{j=0}^{n-t-1} {}_jE_{x+t+1}\, U_{t+1+j}}{\sum_{j=0}^{n-t-2} {}_jE_{x+t+1}\, P_{t+1+j}}.$$

Multiplying both sides by the denominators and cancelling the identical terms on both sides, we get

$$\sum_{j=0}^{n-t-2} {}_jE_{x+t+1}\, U_t P_{t+1+j} \le \sum_{j=0}^{n-t-1} {}_jE_{x+t+1}\, P_t\, U_{t+1+j},$$

which is equivalent to

$$0 \le \sum_{j=0}^{n-t-2} {}_jE_{x+t+1}\, (P_t\, U_{t+1+j} - U_t\, P_{t+1+j}) + {}_{n-t-1}E_{x+t+1}\, P_t U_n. \tag{15}$$

Now, if $j \le n - t - 2$, then $t + 1 + j \le n - 1$. Thus from (8) and (9), we get

$$P_t \ge P_{t+1+j}$$

and

$$U_{t+1+j} \ge U_t.$$

Thus,

$$P_t\, U_{t+1+j} - U_t\, P_{t+1+j} \ge 0 \ \text{ for } \ 0 \le j \le n - t - 2.$$

Furthermore,

$$U_n \ge 0, \ P_t > 0$$

and

$${}_jE_{x+t+1} \ge 0 \ \text{ for } \ 0 \le j \le n - t - 2.$$

Thus, (15) is true which proves (13). ■

COROLLARY 1.1. *Under the conditions of Theorem 1.1,* ${}_tV \geq 0$, $t = 0,1,2,3,\ldots,n$.

Proof: Since the probability of a death benefit payment at time $t = 0$ is zero, we have

$$ {}_0V = {}_0V^{retro} = 0. $$

Thus, we also have

$$ r(0) = 0. $$

So, using (10) of Theorem 1.1, we get

$$ 0 \leq r(t) = \frac{{}_tV}{\sum_{k=0}^{n-t-1} {}_kE_{x+t}\, P_{t+k}}. $$

Therefore,

$$ 0 \leq {}_tV, \; t = 1,2,\ldots,n-1. $$

Furthermore, since there are no premium payments after time $t = n$, we have

$$ {}_nV = {}_nV^{prosp} = U_n \geq 0. \blacksquare $$

Let us see some applications of the theorem to special types of life insurances.

It is reasonable to assume that the probabilities ${}_hq_x$ are increasing in x after a certain age (say x_0) for any $h > 0$. For example, in the table of Appendix 2, we can see that q_x is increasing if $x \geq 10$. Although the table only contains the values of ${}_hq_x$ for integer x and $h = 1$, we can assume that the monotonic property is satisfied for any h and x. Note that if ${}_hq_x$ is increasing in x for any $h > 0$ then ${}_hp_x = 1 - {}_hq_x$ is decreasing in x. We will assume that the policies are issued to people above the age of x_0.

If the premiums are payable as a level annuity-due of P per annum during the term of the insurance then we have

$$ P_0 = P_1 = P_2 = \ldots = P. $$

If the installments of the premium form an annuity-due of P per annum payable $p$thly then we get

$$P_k = \frac{P}{p} \sum_{\ell=0}^{p-1} v^{\frac{\ell}{p}} \, {}_{\frac{\ell}{p}}p_{x+k} \, .$$

Now, ${}_{\frac{\ell}{p}}p_{x+k}$ is decreasing in k for any fixed ℓ, so we get $P_k \geq P_{k+1}$.

If the premium is paid continuously at a rate of P per annum, then

$$P_k = P \int_0^1 v^t \, {}_tp_{x+k} \, dt,$$

and since the term ${}_tp_{x+k}$ is decreasing in k for any fixed t, we get $P_k \geq P_{k+1}$. Hence, in all three cases (8) of Theorem 1.1 is satisfied.

Consider a life insurance with a fixed death benefit of \$1. If the death benefit is payable at the end of the year of death, we have

$$U_k = v \, q_{x+k} \, .$$

Now, q_{x+k} is increasing in k, so $U_k \leq U_{k+1}$.

If the death benefit is payable at the moment of death, we get

$$U_k = \int_0^1 v^t \, {}_tp_{x+k} \, \mu_{x+k+t} \, dt.$$

Let us integrate by parts choosing

$$v(t) = v^t$$

and

$$w(t) = -{}_t\, p_{x+k}.$$

Then,

$$\frac{d}{dt} v(t) = \log v \cdot v^t$$

and from (40) of Section 2.2, we get

$$\frac{d}{dt} w(t) = {}_tp_{x+k} \cdot \mu_{x+k+t}.$$

Thus,

$$U_k = -v^t\, {}_tp_{x+k}\Big|_{t=0}^{1} + \log v \int_0^1 v^t\, {}_tp_{x+k}\, dt$$

$$= 1 - v\, p_{x+k} + \log v \int_0^1 v^t\, {}_tp_{x+k}\, dt$$

$$= 1 - \left(v\, p_{x+k} + \log(1+i) \int_0^1 v^t\, {}_tp_{x+k}\, dt\right).$$

Now, p_{x+k} is decreasing in k and ${}_tp_{x+k}$ is decreasing in k for any fixed t. Therefore, U_k is increasing in k: $U_k \le U_{k+1}$. So, (9) of Theorem 1.1 is satisfied for both types of death benefit payments.

Applying Corollary 1.1, it follows from the above considerations that the reserve is never negative for temporary and whole life and endowment insurances with constant benefits if the premium payments form a level annuity-due or a continuous payment stream with a constant rate of payment. For example, ${}_tV_x$, ${}_tV^{\,1}_{x:\overline{n}|}$, ${}_tV^{(p)}(A_x)$, ${}_tV^{(p)}(A^{\,1}_{x:\overline{n}|})$, ${}_t\overline{V}(\overline{A}_x)$, ${}_t\overline{V}(\overline{A}^{\,1}_{x:\overline{n}|})$, ${}_tV_{x:\overline{n}|}$ and ${}_t\overline{V}(\overline{A}_{x:\overline{n}|})$ are nonnegative.

If the death benefits increase with time then (9) of Theorem 1.1 is even more satisfied, so the reserve is nonnegative again. For example, ${}_tV(IA_x)$ and ${}_tV(IA^{\,1}_{x:\overline{n}|})$ are nonnegative.

An n year term certain insurance requires a similar consideration. We have shown that for the purpose of reserve calculation, the payment of \$1 at the end of year n has to be replaced by a death benefits of v^{n-k} at the end of year k, if death occurs in the year k $(k = 1,2,\ldots,n)$. There is also a survival benefit of \$1 at the end of year n. Since $v^{n-1} \le v^{n-2} \ldots \le v \le 1$, the death benefits form an increasing sequence. This is combined with a survival benefit of \$1, payable at the end of year n. Therefore, whether the premiums form a level annuity-due or a continuous annuity payable at a constant rate, the reserves are nonnegative.

Before we stated Theorem 1.1, we showed why we did not expect any negative reserves to emerge in the case of a whole life insurance with a constant death benefit. We have also seen that if the death benefits increase with time, the reserves cannot be negative either. However, if the death benefits decrease with time, the level annual premium may not be sufficient to cover the higher benefit payments in the early years, so negative reserves may emerge.

Let us see some examples.

Consider an n year term insurance issued to a life aged x that pays an amount of $n - k + 1$ at the end of year of death, if death occurs in year k $(1 \le k \le n)$. That is, the death benefit is n at the end of year 1, $n - 1$ at the

end of year 2,..., and 1 at the end of year n. This insurance can be expressed as an $n+1$ year term insurance, with a constant death benefit of $n+1$ minus an $n+1$ year varying insurance with a death benefit of 1 in year 1 rising to $n+1$ in year $n+1$. Thus, if P denotes the annual premium payable yearly in advance throughout the duration of the insurance, we get

$$P\, \ddot{a}_{x:\overline{n}|} = (n+1)\, A^{\;1}_{x:\overline{n+1}|} - (IA)^{\;1}_{x:\overline{n+1}|},$$

hence

$$P = \frac{(n+1)\, A^{\;1}_{x:\overline{n+1}|} - (IA)^{\;1}_{x:\overline{n+1}|}}{\ddot{a}_{x:\overline{n}|}}.$$

The reserve at duration t is

$$\begin{aligned} {}_tV = {}_tV^{prosp} &= (n-t+1)A^{\;1}_{x+t:\overline{n-t+1}|} - (IA)^{\;1}_{x+t:\overline{n-t+1}|} - P\, \ddot{a}_{x+t:\overline{n-t}|} \\ &= (n-t+1)A^{\;1}_{x+t:\overline{n-t+1}|} - (IA)^{\;1}_{x+t:\overline{n-t+1}|} \\ &\quad - ((n+1)A^{\;1}_{x:\overline{n+1}|} - (IA)^{\;1}_{x:\overline{n+1}|}) \frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}}. \end{aligned}$$

We worked out this problem with $x = 30$, $n = 25$ and 6% annual interest rate. The result is shown in Table 1. The letter m stands for the length of the premium payment period, P for the annual premium corresponding to this payment period, and $V(t)$ for the reserve ${}_tV$ at duration t.

We can see that if $m = n = 25$, the reserve is negative for every t between 1 and 24. How can we get rid of the negative reserves?

The usual solution is to shorten the premium payment period. We have already seen earlier in this section that the shortening of the payment period increases the reserve at any duration. So, we can hope that if the payment term is short enough, the negative reserves will disappear. Actually, in the extreme case of a single premium payable at the start of the insurance, the reserve is always nonnegative.

If ${}_mP$ denotes the annual premium whose payment is limited to m years, we have

$${}_mP\, \ddot{a}_{x:\overline{m}|} = (n+1)A^{\;1}_{x:\overline{n+1}|} - (IA)^{\;1}_{x:\overline{n+1}|},$$

and therefore

$${}_mP = \frac{(n+1)A^{\;1}_{x:\overline{n+1}|} - (IA)^{\;1}_{x:\overline{n+1}|}}{\ddot{a}_{x:\overline{m}|}}.$$

TABLE 1

Elimination of negative reserves through shortening of the premium period (decreasing n-year term insurance)

$x = 30$
$n = 25$

t	$m = 25$ $P = 0.0357231$ $V(t)$	$m = 24$ $P = 0.0363441$ $V(t)$	$m = 23$ $P = 0.0370316$ $V(t)$	$m = 22$ $P = 0.0377948$ $V(t)$
0	0.0000000	0.0000000	0.0000000	0.0000000
1	-0.0003549	0.0003044	0.0010343	0.0018446
2	-0.0011266	0.0002326	0.0017376	0.0034081
3	-0.0023523	-0.0002498	0.0020783	0.0046625
4	-0.0040716	-0.0011794	0.0020228	0.0055775
5	-0.0063208	-0.0025898	0.0015413	0.0061269
6	-0.0091230	-0.0045006	0.0006174	0.0062987
7	-0.0124955	-0.0069256	-0.0007585	0.0060873
8	-0.0164391	-0.0098617	-0.0025792	0.0055048
9	-0.0209430	-0.0132942	-0.0048253	0.0045756
10	-0.0259727	-0.0171839	-0.0074527	0.0033493
11	-0.0314705	-0.0214683	-0.0103937	0.0018997
12	-0.0373477	-0.0260534	-0.0135482	0.0003332
13	-0.0434843	-0.0308136	-0.0167844	-0.0012112
14	-0.0497106	-0.0355728	-0.0199191	-0.0025427
15	-0.0558101	-0.0401075	-0.0227213	-0.0034217
16	-0.0615040	-0.0441316	-0.0248965	-0.0035446
17	-0.0664417	-0.0472860	-0.0260766	-0.0025330
18	-0.0701863	-0.0491249	-0.0258053	0.0000806
19	-0.0722005	-0.0491007	-0.0235242	0.0048671
20	-0.0718282	-0.0465460	-0.0185532	0.0125203
21	-0.0682754	-0.0406546	-0.0100724	0.0238755
22	-0.0605828	-0.0304530	0.0029072	0.0399388
23	-0.0476007	-0.0147759	0.0215681	0.0215681
24	-0.0279530	0.0077701	0.0077701	0.0077701
25	0.0000000	0.0000000	0.0000000	0.0000000

t	$m = 21$ $P = 0.0386446$ $V(t)$	$m = 20$ $P = 0.0395937$ $V(t)$	$m = 19$ $P = 0.04065181$ $V(t)$
0	0.0000000	0.0000000	0.0000000
1	0.0027467	0.0037543	0.0048843
2	0.0052680	0.0073456	0.0096754
3	0.0075395	0.0107534	0.0143573
4	0.0095349	0.0139557	0.0189129
5	0.0112323	0.0169352	0.0233303
6	0.0126238	0.0196893	0.0276122
7	0.0137089	0.0222227	0.0317697
8	0.0145049	0.0245586	0.0358324
9	0.0150419	0.0267333	0.0398436
10	0.0153755	0.0288095	0.0438737
11	0.0155862	0.0308749	0.0480190
12	0.0157876	0.0330512	0.0524099
13	0.0161267	0.0354943	0.0572123
14	0.0168028	0.0384130	0.0626457
15	0.0180648	0.0420667	0.0689814
16	0.0202268	0.0467811	0.0765579
17	0.0236785	0.0529585	0.0857918
18	0.0288999	0.0610929	0.0971927
19	0.0364757	0.0717845	0.1113783
20	0.0471150	0.0857595	0.0857595
21	0.0616704	0.0616704	0.0616704
22	0.0399388	0.0399388	0.0399388
23	0.0215681	0.0215681	0.0215681
24	0.0077701	0.0077701	0.0077701
25	0.0000000	0.0000000	0.0000000

Now, if $t \leq m,$ we obtain

$$\begin{aligned} {}_tV &= (n - t + 1)A^{1}_{x+t:\overline{n-t+1}|} - (IA)^{1}_{x+t:\overline{n-t+1}|} - {}_mP\ddot{a}_{x+t:\overline{m-t}|} \\ &= (n - t + 1)A^{1}_{x+t:\overline{n-t+1}|} - (IA)^{1}_{x+t:\overline{n-t+1}|} - ((n + 1)A^{1}_{x:\overline{n+1}|} - (IA)^{1}_{x:\overline{n+1}|})\frac{\ddot{a}_{x+t:\overline{m-t}|}}{\ddot{a}_{x:\overline{m}|}} \end{aligned}$$

and if $t > m,$ we get

$${}_tV = (n - t + 1)A^{1}_{x+t:\overline{n-t+1}|} - (IA)^{1}_{x+t:\overline{n-t+1}|}.$$

We can see from Table 1, that if we decrease the premium payment period by one year, that is, $m = 24,$ some negative reserves disappear, but some others are still there. We have to go down to a 21 year premium payment term in order to remove all the negative reserves. Of course, any premium payment term shorter than 21 will also do.

Remember that when we determined the reserve for an n year family income benefit of $\$1$ per annum, we showed that from the point of view of the reserves, this insurance is equivalent to an n year term insurance with a death benefit of $\ddot{a}_{\overline{n-k+1}|}$ payable at the end of year $k,$ if death occurs in year k $(1 \leq k \leq n)$. Note that as k increases, $\ddot{a}_{\overline{n-k+1}|}$ decreases. So, we are again dealing with an insurance whose death benefit is decreasing with time. Therefore, it is possible that negative reserves emerge here, too. The annual premium P for this insurance is

$$P = \frac{a_{\overline{n}|} - a_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$$

and the reserve at duration t can be expressed as

$${}_tV = a_{\overline{n-t}|} - a_{x+t:\overline{n-t}|} - P\,\ddot{a}_{x+t:\overline{n-t}|}.$$

Using $x = 20,$ $n = 15,$ and a 6% annual interest rate, we obtain the reserves given in Table 2. If the premium payment period is of length 15 $(m = 15)$, all the reserves between times $t = 1$ and $t = 14$ are negative.

In order to avoid negative reserves, we will again shorten the premium payment period.

If ${}_mP$ denotes the annual premium payable for m years only, we have

$${}_mP = \frac{a_{\overline{n}|} - a_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}}.$$

If $t \leq m,$ we get

$${}_tV = a_{\overline{n-t}|} - a_{x+t:\overline{n-t}|} - P\,\ddot{a}_{x+t:\overline{n-t}|}$$

and if $t > m$, we obtain

$$_tV = a_{\overline{n-t}|} - a_{x+t:\overline{n-t}|}\,.$$

Table 2 shows that the premium payment term has to be limited to 10 years or less in order to get rid of the negative reserves.

If the premium and benefit payments are always made at the beginning or at the end of the years during the term of the insurance, there is a simple recursive relationship between the reserves of successive years. Let us consider the following model.

An insurance is taken out at time $t_0 = 0$ at the age of x. The term of the insurance is N years, where N is a positive integer or infinity. If the insured survives to time t (where t is an integer such that $0 \le t \le N$), a survival benefit of B_t and a premium of P_t become payable at time t. If the insured dies between times $t - 1$ and t (where t is an integer such that $1 \le t \le N$), a death benefit of S_t is paid at time t.

The amounts $B_t, P_t,$ and S_t can be zero. This way, the model describes a wide range of insurances: pure endowments, annuities, temporary and whole life insurances, or endowment insurances with single or annual premium payments. Of course, if the insurance contains death benefit payments, they have to take place at the end of the year of death in order to fit the model. For example, an n year annuity-immediate of $1 per annum, with a single premium, can be described by $N = n$,

$$P_t = \begin{cases} \dfrac{1}{\ddot{a}_{x:\overline{n}|}}, & \text{if } t = 0 \\ 0, & \text{if } 1 \le t \le N, \end{cases}$$

$$B_t = \begin{cases} 0, & \text{if } t = 0 \\ 1, & \text{if } 0 < t \le N, \end{cases}$$

and

$$S_t = 0, \text{ if } 1 \le t \le N.$$

A whole life insurance of $1 with annual premium payments can be modeled by

$$N\text{: infinity},$$

$$P_t = \frac{A_x}{\ddot{a}_x}, \text{ if } 0 \le t$$

$$B_t = 0, \text{ if } 0 \le t$$

TABLE 2

Elimination of negative reserves through shortening of the premium payment period (n-year family income benefit)

$x = 20$
$n = 15$

t	$m = 15$ $P = 0.0079617$ $V(t)$	$m = 14$ $P = 0.0083148$ $V(t)$	$m = 13$ $P = 0.0087257$ $V(t)$	$m = 12$ $P = 0.0092089$ $V(t)$
0	0.0000000	0.0000000	0.0000000	0.0000000
1	-0.0021663	-0.0017916	-0.0013556	-0.0008429
2	-0.0043233	-0.0035511	-0.0026525	-0.0015957
3	-0.0064408	-0.0052468	-0.0038571	-0.0022229
4	-0.0084846	-0.0068428	-0.0049320	-0.0026850
5	-0.0104061	-0.0082890	-0.0058251	-0.0029277
6	-0.0121473	-0.0095257	-0.0064748	-0.0028869
7	-0.0136357	-0.0104785	-0.0068042	-0.0024834
8	-0.0147805	-0.0110547	-0.0067186	-0.0016195
9	-0.0154716	-0.0111420	-0.0061032	-0.0001778
10	-0.0155722	-0.0106014	-0.0048163	0.0019868
11	-0.0149163	-0.0092643	-0.0026866	0.0050488
12	-0.0133030	-0.0069274	0.0004925	0.0092182
13	-0.0104865	-0.0033420	0.0049728	0.0049728
14	-0.0061711	0.0017906	0.0017906	0.0017906
15	0.0000000	0.0000000	0.0000000	0.0000000

t	$m = 11$ $P = 0.0097842$ $V(t)$	$m = 10$ $P = 0.0104791$ $V(t)$	$m = 9$ $P = 0.0113337$ $V(t)$
0	0.0000000	0.0000000	0.0000000
1	-0.0002325	0.0005049	0.0014117
2	-0.0003375	0.0011824	0.0030514
3	-0.0002773	0.0020730	0.0049631
4	-0.0000099	0.0032217	0.0071956
5	0.0005218	0.0046889	0.0098130
6	0.0013845	0.0065446	0.0128897
7	0.0026606	0.0088749	0.0165163
8	0.0044509	0.0117844	0.0208021
9	0.0068765	0.0153985	0.0258776
10	0.0100859	0.0198701	0.0198701
11	0.0142577	0.0142577	0.0142577
12	0.0092182	0.0092182	0.0092182
13	0.0049728	0.0049728	0.0049728
14	0.0017906	0.0017906	0.0017906
15	0.0000000	0.0000000	0.0000000

and

$$S_t = 1, \text{ if } 1 \leq t.$$

If we consider an n year endowment insurance of \$1, with annual premiums, we get the model

$$N = n,$$

$$P_t = \begin{cases} \dfrac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}, & \text{if } 0 \leq t \leq N - 1 \\ 0, & \text{if } t = N, \end{cases}$$

$$B_t = \begin{cases} 0, & \text{if } 0 \leq t \leq N - 1 \\ 1, & \text{if } t = N, \end{cases}$$

and

$$S_t = 1, \text{ if } 1 \leq t \leq N.$$

Let us now return to the general model. The reserve at time $t_0 = 0$ is zero: ${}_0V = 0$. If $0 \leq t \leq N - 1$, the relationship between ${}_tV$ and ${}_{t+1}V$ can be obtained as follows. Considering ${}_tV$ from a prospective point of view, it is the expected present value of the cash flow after time t. Let us split this cash flow into two parts. The present value at time t of the cash flow between t and $t + 1$ is

$$B_t - P_t + v\, q_{x+t}\, S_{t+1} = B_t - P_t + \frac{C_{x+t}}{D_{x+t}}\, S_{t+1}.$$

On the other hand, the present value at time $t + 1$ of the cash flow after time $t + 1$ equals ${}_{t+1}V$. So, using (20) of Section 3.1, we find that the present value of this cash flow at time t is

$${}_1E_{x+t}\; {}_{t+1}V = v\, p_{x+t}\; {}_{t+1}V.$$

Therefore, we get

$${}_tV = B_t - P_t + v\, q_{x+t}\, S_{t+1} + v\, p_{x+t}\; {}_{t+1}V. \tag{16}$$

Thus,

$${}_{t+1}V = \frac{({}_tV - B_t + P_t) - v\, q_{x+t}\, S_{t+1}}{v p_{x+t}}.$$

So, introducing the functions

$$u_x = \frac{1}{v\, p_x}$$

and

$$k_x = \frac{q_x}{p_x},$$

we can write

$$_{t+1}V = ({}_tV - B_t + P_t)\, u_{x+t} - S_{t+1}\, k_{x+t} \quad \text{for } t = 0,1,\ldots,N-1. \tag{17}$$

The functions u_x and k_x are called the Fackler valuation functions. They can easily be expressed in terms of the commutation functions:

$$u_x = \frac{1}{v p_x} = \frac{v^x\, \ell_x}{v^{x+1}\, \ell_{x+1}} = \frac{D_x}{D_{x+1}} \tag{18}$$

and

$$k_x = \frac{q_x}{p_x} = \frac{d_x}{\ell_x} \cdot \frac{\ell_x}{\ell_{x+1}} = \frac{v^{x+1}\, d_x}{v^{x+1}\, \ell_{x+1}} = \frac{C_x}{D_{x+1}}. \tag{19}$$

Thus, (17) can be rewritten as

$$_{t+1}V = \frac{({}_tV - B_t + P_t)\, D_{x+t} - S_{t+1}\, C_{x+t}}{D_{x+t+1}} \quad \text{for } t = 0,1,\ldots,N-1. \tag{20}$$

If we want to know the reserves in successive years, it is usually more convenient to use the recursive relationship (17) than to compute the reserve directly at every duration. For example, starting out with $_0V = 0$, using (17) we obtain the values of $_1V, _2V, \ldots,$ in turn. If N is a finite number, we have $_NV = {}_NV^{prosp} = B_N - P_N$. Thus, if we compute all $_tV$ for $t = 0,\ldots,N$ from the recursive relationship, we should obtain a value for $_NV$ equaling $B_N - P_N$ (of course, allowing for round-off errors). If this is not the case, something has gone wrong with the computations.

Since applying a recursive relationship has the disadvantage that the round off errors get accumulated, it is advisable to use more decimals in the computations than usual. However, if we use a computer, recursive formulas are very easy to program and the calculations can be done with high accuracy.

EXAMPLE 1.10. A 10 year term insurance of $5000 issued to a life aged 45 is purchased by annual premiums. Find the reserve at the end of year 4, and based on this, derive the reserve at the end of year 5, using a recursive formula. Use a 6% annual rate of interest.

Solution: The annual premium is

$$\begin{aligned} P &= 5000 \frac{A_{45:\overline{10}|}}{\ddot{a}_{45:\overline{10}|}} \\ &= 5000 \frac{M_{45} - M_{55}}{N_{45} - N_{55}} \\ &= 5000 \frac{1339.5427 - 1069.6405}{93953.92 - 43031.29} \\ &= \$26.5012. \end{aligned}$$

Thus, the reserve at the end of year 4 is

$$\begin{aligned} {}_4V &= 5000\, A_{49:\overline{6}|} - 26.5012\, \ddot{a}_{49:\overline{6}|} \\ &= \frac{5000(M_{49} - M_{55}) - 26.5012\,(N_{49} - N_{55})}{D_{49}} \\ &= \frac{5000\,(1236.8813 - 1069.6405) - 26.5012(69646.60 - 43031.29)}{5179.14} \\ &= \$25.2680. \end{aligned}$$

Furthermore, we have

$$B_4 = 0,$$

$$P_4 = P = 26.5012$$

and

$$S_5 = 5000.$$

Thus, using (20), the reserve at the end of year 5 is

$$\begin{aligned} {}_5V &= \frac{({}_4V - B_4 + P_4)\, D_{49} - 5000\, C_{49}}{D_{50}} \\ &= \frac{(25.2680 + 26.5012)5179.14 - 5000 \times 26.6857}{4859.30} \\ &= \$27.7183. \end{aligned}$$

EXAMPLE 1.11. The premium for a 20 year endowment of $3000 issued to a life aged 50, is payable annually. Using the recursive formula, find the reserves at duration $t = 0,1,2,\ldots,20$, based on a 6% annual interest rate.

Solution: The annual premium for the insurance is

$$
\begin{aligned}
P &= 3000 \frac{A_{50:\overline{20}|}}{\ddot{a}_{50:\overline{20}|}} \\
&= 3000 \frac{M_{50} - M_{70} + D_{70}}{N_{50} - N_{70}} \\
&= 3000 \frac{1210.1957 - 576.7113 + 1119.94}{64467.45 - 9597.05} \\
&= \$95.8673.
\end{aligned}
$$

So, we have

$$N = 10,$$

$$P_t = \begin{cases} 95.8673, & \text{if } 0 \le t \le 19 \\ 0, & \text{if } t = 20, \end{cases}$$

$$B_t = \begin{cases} 0, & \text{if } 0 \le t \le 19 \\ 3000, & \text{if } t = 20, \end{cases}$$

and

$$S_t = 3000, \text{ if } 1 \le t \le 20.$$

Furthermore,

$$u_{x+t} = u_{50+t} = \frac{D_{50+t}}{D_{51+t}}$$

and

$$k_{x+t} = k_{50+t} = \frac{C_{50+t}}{D_{51+t}}, \text{ for } t = 0,1,\ldots,19.$$

The results of the computations are given in Table 3.

The notations $V(t)$, $B(t)$, $P(t)$, and $S(t)$ stand for ${}_tV$, B_t, P_t, and S_t, respectively.

As a quick check of the computations, we can compare the value of ${}_{20}V$ standing in the table with ${}_{20}B$ - P_{20}. They are equal, their common value is \$3000.

We can see that the application of formula (17) is simpler than using the formula

TABLE 3
Calculation of reserves using the recursive relationship
(n-year endowment insurance)

$x = 50$
$n = 20$

$P = 95.8673$

t	$V(t)$	$B(t)$	$P(t)$	$u(x+t)$	$S(t)$	$k(x+t)$
0	0.0000	0	95.8673	1.0663125		0.0059552
1	84.3588	0	95.8673	1.0668514	3000	0.0064636
2	172.8837	0	95.8673	1.0674427	3000	0.0070214
3	265.8120	0	95.8673	1.0680913	3000	0.0076333
4	363.4066	0	95.8673	1.0688030	3000	0.0083047
5	465.9591	0	95.8673	1.0695841	3000	0.0090416
6	573.7958	0	95.8673	1.0704409	3000	0.0098499
7	687.2851	0	95.8673	1.0713813	3000	0.0107371
8	806.8436	0	95.8673	1.0724134	3000	0.0117108
9	932.9470	0	95.8673	1.0735461	3000	0.0127793
10	1066.1415	0	95.8673	1.0747896	3000	0.0139525
11	1207.0576	0	95.8673	1.0761546	3000	0.0152402
12	1356.4280	0	95.8673	1.0776534	3000	0.0166541
13	1515.1085	0	95.8673	1.0792991	3000	0.0182067
14	1684.1046	0	95.8673	1.0811065	3000	0.0199118
15	1864.6037	0	95.8673	1.0830918	3000	0.0217847
16	2058.0159	0	95.8673	1.0852729	3000	0.0238424
17	2266.0240	0	95.8673	1.0876692	3000	0.0261030
18	2490.6473	0	95.8673	1.0903031	3000	0.0285878
19	2734.3214	0	95.8673	1.0931982	3000	0.0313191
20	3000.0000	3000	0.0000		3000	

$$
{}_tV^{prosp} = A_{50+t:\overline{20-t}|} - P\,\ddot{a}_{50+t:\overline{20-t}|} = \frac{M_{50+t} - M_{70} + D_{70} - P(N_{50+t} - N_{70})}{D_{50+t}}
$$

or

$$
{}_tV^{retro} = \frac{1}{{}_tE_{50}}\left(A^{1}_{50:\overline{t}|} - P\,\ddot{a}_{50+\overline{t}|}\right) = \frac{M_{50} - M_{50+t} - P(N_{50} - N_{50+t})}{D_{50+t}}
$$

over and over again. The recursive method can also be used if the death benefit payments extend beyond the end of the year of death. In this case, the death benefit payments have to be replaced by their capitalized values payable at the end of the year of death.

Thus, an n year annuity-immediate of \$1 per annum, with guaranteed payments in the first m years, can be described by

$$N = n,$$

$$P_t = \begin{cases} a_{\overline{m}|} + {}_{m|}a_{x:\overline{n-m}|}, & \text{if } t = 0 \\ 0, & \text{if } 1 \le t \le n, \end{cases}$$

$$B_t = \begin{cases} 0, & \text{if } t = 0 \\ 1, & \text{if } 1 \le t \le n, \end{cases}$$

and

$$S_t = \begin{cases} \ddot{a}_{\overline{m-t+1}|}, & \text{if } 1 \le t \le m \\ 0, & \text{if } m + 1 \le t \le n. \end{cases}$$

An n year family income benefit of \$1 per annum can be modeled by

$$N = n,$$

$$B_t = 0, \text{ if } 0 \le t \le n,$$

and

$$S_t = \ddot{a}_{\overline{n-t+1}|}, \text{ if } 1 \le t \le n.$$

The expression for the premium depends on the premium paying method. If a single premium is paid then,

$$P_t = \begin{cases} a_{\overline{n}|} - a_{x:\overline{n}|}, & \text{if } t = 0 \\ 0, & \text{if } 1 \le t \le n \end{cases}$$

and if the premium is paid annually for m years, we get

$$P_t = \begin{cases} \dfrac{a_{\overline{n}|} - a_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}}, & \text{if } 0 \le t \le m - 1 \\ 0, & \text{if } m \le t \le n. \end{cases}$$

premium payments we get the model

$$N = n,$$

$$B_t = \begin{cases} 0, & \text{if } 0 \le t \le n - 1 \\ 1, & \text{if } t = n, \end{cases}$$

and

$$S_t = v^{n-t}, \text{ if } 1 \le t \le n.$$

Furthermore, if the premium is paid annually for m years, we get

$$P_t = \begin{cases} \dfrac{v^{n-t}}{\ddot{a}_{x:\overline{m}|}}, & \text{if } 0 \le t \le m - 1 \\ 0, & \text{if } m \le t \le n. \end{cases}$$

EXAMPLE 1.12. A 20 year annuity-immediate of \$1000 per annum, with guaranteed payments in the first 5 years, is purchased for a life aged 40 by a single premium. Using the recursive formula, find the reserves at duration $t = 0,1,2,\ldots,20$, based on a 6% annual rate of interest.

Solution: The single premium for the insurance is

$$\begin{aligned} P &= 1000\,(a_{\overline{5}|} + {}_{5|}a_{40:\overline{15}|}) \\ &= 1000\left(a_{\overline{5}|} + \frac{N_{46} - N_{61}}{D_{40}}\right) \\ &= 1000\left(4.2124 + \frac{87296.23 - 25182.39}{9054.46}\right) \\ &= \$11072.39. \end{aligned}$$

Thus, we have

$$N = 20$$

$$P_t = \begin{cases} 11072.39, & \text{if } t = 0 \\ 0, & \text{if } 1 \le t \le 20, \end{cases}$$

$$B_t = \begin{cases} 0, & \text{if } t = 0 \\ 1000, & \text{if } 1 \le t \le 20 \end{cases},$$

and

$$S_t = \begin{cases} 1000\, \ddot{a}_{\overline{6-t}|}, & \text{if } 1 \le t \le 5 \\ 0, & \text{if } 6 \le t \le 20. \end{cases}$$

Furthermore,

$$u_{x+t} = u_{40+t} = \frac{D_{40+t}}{D_{41+t}}, \text{ for } t = 0,1,\ldots,19$$

and

$$k_{x+t} = k_{40+t} = \frac{C_{40+t}}{D_{41+t}}, \text{ for } t = 0,1,\ldots,19.$$

The results of the computation are given in Table 4.

TABLE 4
Calculation of reserves using the recursive relationship
(n-year annuity-immediate with guaranteed payments in the first m years)

$x = 40$
$n = 20$
$m = 5$

$P = 11072.3897$

t	$V(t)$	$B(t)$	$P(t)$	$u(x+t)$	$S(t)$	$k(x+t)$
0	0.0000	0	11072.389	1.0629563		0.0027890
1	11757.0136	1000	7	1.0631701	4465.1056	0.0029906
2	11425.5503	1000	0	1.0634048	3673.0119	0.0032121
3	11077.4789	1000	0	1.0636619	2833.3927	0.0034546
4	10712.3166	1000	0	1.0639440	1943.3962	0.0037207
5	10329.6400	1000	0	1.0642534	1000.0000	0.0040126
6	9929.1010	1000	0	1.0645928	0	0.0043328
7	9505.8567	1000	0	1.0649650	0	0.0046840
8	9058.4400	1000	0	1.0653733	0	0.0050692
9	8585.2472	1000	0	1.0658211	0	0.0054916
10	8084.5167	1000	0	1.0663125	0	0.0059552
11	7554.3089	1000	0	1.0668514	0	0.0064636
12	6992.4736	1000	0	1.0674427	0	0.0070214
13	6396.6223	1000	0	1.0680913	0	0.0076333
14	5764.0853	1000	0	1.0688030	0	0.0083047
15	5091.8687	1000	0	1.0695841	0	0.0090416
16	4376.5976	1000	0	1.0704409	0	0.0098499
17	3614.4481	1000	0	1.0713813	0	0.0107371
18	2801.0707	1000	0	1.0724134	0	0.0117108
19	1931.4924	1000	0	1.0735461	0	0.0127793
20	1000.0000	1000	0		0	
			0			

PROBLEMS

1.1. A 20 year pure endowment insurance of \$2000, on a life aged 30, is purchased by a single premium. Find the expressions for the prospective and retrospective reserves at the end of each policy year. Evaluate the reserves numerically at the end of year 6 and year 18. Use a 6% annual rate of interest.

1.2. The premium for a whole life insurance of \$8000, issued to a life aged 40 is payable at the commencement of the insurance. Obtain the expressions for the prospective and retrospective reserves at the end of each policy year. Calculate the reserves numerically at the end of year 1 and year 5. Base the computations on a 6% annual rate of interest.

1.3. The death benefit of a 15 year endowment insurance of \$4000, issued to a life aged 25 is payable at the moment of death. The premium is payable at the commencement of the insurance. Derive the expressions for the prospective and retrospective reserves at the end of each policy year. Based on a 6% annual rate of interest, determine the numerical value of the reserves at the end of year 5 and year 8.

1.4. A 10 year annuity of \$300 per annum payable yearly in advance is purchased for a life aged 45 by a single premium. Determine the expressions for the prospective and retrospective reserves at the end of each policy year. Based on a 6% annual rate of interest, evaluate the reserves numerically at the end of year 6 and year 10.

1.5. A 10 year family income benefit of \$6000 per annum is purchased for a life aged 50 by a single premium. Derive the expressions for the prospective and retrospective reserves at the end of each policy year and calculate them numerically at the end of year 5. Use a 6% annual rate of interest.

1.6. A 20 year annuity-immediate of \$1000 per annum, with the first 15 payments guaranteed, is purchased for a life aged 40 by a single premium. Find the expressions for the prospective and retrospective reserves at the end of each policy year. Based on a 6% annual rate of interest, evaluate the reserves numerically at the end of years 10, 15, and 18.

1.7. Show that

a) ${}_tV_x = 1 - (1 - {}_1V_x)\,(1 - {}_1V_{x+1}) \cdots (1 - {}_1V_{x+t-1})$

and

b) ${}_tV_{x:\overline{n}|} = 1 - (1 - {}_1V_{x:\overline{n}|})\,(1 - {}_1V_{x+1:\overline{n-1}|}) \cdots (1 - {}_1V_{x+t-1:\overline{n-t+1}|})$.

1.8. Based on a 6% annual rate of interest, obtain

a) ${}_3V^{1}_{35:\overline{10}|}$

b) ${}_{20}V_{40}$

c) ${}_7\underline{V}_{45:\overline{15}|}$

d) ${}_5V\,(A_{30:\overline{20}|})$

e) ${}_{10}\overline{V}\,(\overline{A}_{50})$

f) ${}_{14}V^{(4)}{}^{1}_{20:\overline{35}|}$

g) ${}^{5}_{2}V^{1}_{60:\overline{20}|}$

h) ${}^{10}_{5}V^{(12)}_{30}$.

1.9. Repeat Problem 1.1 if

a) an annual premium is payable for the whole term of the insurance.

b) an annual premium is payable for a term of 10 years.

1.10. Repeat Problem 1.2 if

a) an annual premium is payable for the whole term of the insurance.

b) an annual premium is payable for a term of 5 years.

1.11. Repeat Problem 1.3 if the premium is payable continuously throughout the term of the insurance.

1.12. A 20 year annuity-due of \$8000 per annum, deferred 5 years is purchased for a life aged 60 by monthly premiums payable in the 5 years of the deferment period. Based on a 6% annual rate of interest, find the reserves at the end of years 3, 5, and 10.

1.13. A 10 year term certain insurance of \$8000 is issued to a life aged 30. Based on a 6% annual rate of interest, find the reserve at the end of year 3, if

a) the premium is payable annually for the whole term of the insurance.

b) the premium is payable annually for a term of 5 years.

1.14. The death benefit of a 20 year term insurance issued to a life aged 40, is \$8000 in the first year and decreases linearly to \$400 in year 20.

a) Evaluate the reserves at the end of each policy year if the premium is payable annually over the whole term of the insurance.

b) Find the longest premium paying period under which the reserves are not negative.

1.15. Assume that the premiums for the family income benefit, given in Problem 1.5, are payable yearly in advance. Determine the longest premium paying period under which the reserves are not negative.

1.16. A 25 year endowment insurance of \$5000 is purchased for a life aged 40 by 10 annual premiums. Using the recursive formula, obtain the reserves at the end of each policy year. Base the computations on a 6% annual interest rate.

1.17. Use the recursive formula to obtain the reserve at the end of years 7 and 8 for the pure endowment insurances defined in Problem 1.9.

1.18. Using the recursive formula, evaluate the reserves at the end of each policy year for the insurance given in Problem 1.6.

5.2. Mortality Profit

Assume we have an insurance policy whose reserve at duration t is ${}_tV$ ($t = 0,1,2,\ldots$). The insurance company has to set aside this amount at time t in order to be able to meet future liabilities. In the course of the following year, premiums will be received and benefit payments will be made. Moreover, if the insured is still alive at time $t+1$, the reserve ${}_{t+1}V$ has to be set aside at duration $t+1$. Thus, at time $t+1$, the company can compute the accumulation of the exact (not stochastic) cash flow consisting of ${}_tV$, the actual premium and benefit payments between t and $t+1$, and possibly ${}_{t+1}V$ if the insured is still alive then. If the accumulated value is positive, the company can consider it as a profit made on the policy during the year. Since this profit is determined by the mortality experience, it is called the mortality profit. If the accumulated value is negative, there is a loss on the policy, called the mortality loss. Since the premiums and the reserves are obtained as expected values of random variables, using the mortality profit or loss as real financial indicators in the books of a company only makes sense if we are dealing with a group of policies. Even then, the expression "mortality surplus" is often preferred to "mortality profit" since the final profit on a policy can only be determined when the policy is terminated.

Let us recall the model introduced at the end of Section 5.1. It consists of premiums P_t and survival benefits B_t payable at $t = 0,1,2,\ldots,N,$ and death benefits S_t payable at $t = 1,2,\ldots,N.$ If an insurance fits this model, simple expressions can be found for the mortality profit or loss.

If the company makes adequate reserves at time $t,$ the money set aside for a policy in force at t is ${}_tV.$ Since the insured is alive at $t,$ the premium P_t and the survival benefits will be paid at that time. Now ${}_tV - B_t + P_t$ will accumulate to $({}_tV - B_t + P_t)(1+i)$ by time $t+1.$

If the insured dies in the time period between t and $t + 1$, a death benefit of S_{t+1} will be paid at time $t + 1$. Thus the mortality profit made on the policy in year $t + 1$ is

$$({}_tV - B_t + P_t)(1 + i) - S_{t+1}. \tag{1}$$

Of course, (1) can be negative, which means a mortality loss has been made on the policy for the year.

On the other hand, if the insured is still alive at time t, the company has to make a reserve of ${}_{t+1}V$ at time $t + 1$. So the mortality profit made on the policy in year $t + 1$ is

$$({}_tV - B_t + P_t)(1 + i) - {}_{t+1}V. \tag{2}$$

Again, if (2) is negative, that means a mortality loss on the policy for the year.

Assume there are n_0 people who are of equal age and who buy the same insurance at the same time. Moreover let n_t denote the number of policyholders still alive at time t ($t = 1,2,\dots,N$). Then the total mortality profit on the group of n_t policies for the year $t + 1$ is

$$\sum_{k=1}^{n_{t+1}} (({}_tV - B_t + P_t)(1 + i) - {}_{t+1}V) + \sum_{k=1}^{n_t - n_{t+1}} (({}_tV - B_t + P_t)(1 + i) - S_{t+1})$$

$$= \sum_{k=1}^{n_t} ({}_tV - B_t + P_t)(1 + i) - \sum_{k=1}^{n_{t+1}} {}_{t+1}V - \sum_{k=1}^{n_t - n_{t+1}} S_{t+1}. \tag{3}$$

In many applications, it is more convenient to express the mortality profit in one of the following ways:

$$\left(\sum_{k=1}^{n_t} {}_tV - \sum_{k=1}^{n_t} B_t + \sum_{k=1}^{n_t} P_t \right)(1 + i) - \sum_{k=1}^{n_{t+1}} {}_{t+1}V - \sum_{k=1}^{n_t - n_{t+1}} S_{t+1} \tag{4}$$

or

$$n_t({}_tV - B_t + P_t)(1 + i) - n_{t+1}\, {}_{t+1}V - (n_t - n_{t+1})S_{t+1}. \tag{5}$$

We can interpret (4) as follows.

Total mortality profit for a year
= (Total reserve at the beginning of the year
- Total survival benefit payments at the beginning of the year
+ Total premium payment at the beginning of the year) $(1 + i)$

- Total reserve at the end of the year
- Total death benefit payment at the end of the year, (6)

where "total" means that all policies in force at the respective time are taken into account.

Using (16) of Section 5.1, we can get another expression for the mortality profit. We can write

$$ {}_tV - B_t + P_t = v\, q_{x+t}\, S_{t+1} + v\, p_{x+t}\, {}_{t+1}V, $$

and hence

$$ ({}_tV - B_t + P_t)(1 + i) = q_{x+t}\, S_{t+1} + p_{x+t}\, {}_{t+1}V. \tag{7} $$

Therefore, (2) and (7) imply that if the insured is alive at time $t + 1$ then the mortality profit on the policy for the year is

$$ q_{x+t}\, S_{t+1} + p_{x+t}\, {}_{t+1}V - {}_{t+1}V = q_{x+t}(S_{t+1} - {}_{t+1}V). \tag{8} $$

Furthermore, (1) and (7) give the following expression for the annual mortality profit on the policy if the insured has died during the year:

$$ \begin{aligned} q_{x+t}\, S_{t+1} + p_{x+t}\, {}_{t+1}V - S_{t+1} &= p_{x+t}({}_{t+1}V - S_{t+1}) \\ &= (1 - q_{x+t})({}_{t+1}V - S_{t+1}) \\ &= q_{x+t}(S_{t+1} - {}_{t+1}V) - (S_{t+1} - {}_{t+1}V). \end{aligned} \tag{9} $$

Therefore, the total mortality profit on the group of n_t policies for the year t is

$$ \begin{aligned} &\sum_{k=1}^{n_{t+1}} q_{x+t}(S_{t+1} - {}_{t+1}V) + \sum_{k=1}^{n_t - n_{t+1}} (q_{x+t}(S_{t+1} - {}_{t+1}V) - (S_{t+1} - {}_{t+1}V)) \\ &\qquad = \sum_{k=1}^{n_t} q_{x+t}(S_{t+1} - {}_{t+1}V) - \sum_{k=1}^{n_t - n_{t+1}} (S_{t+1} - {}_{t+1}V) \\ &\qquad = n_t\, q_{x+t}(S_{t+1} - {}_{t+1}V) - (n_t - n_{t+1})(S_{t+1} - {}_{t+1}V). \end{aligned} \tag{10} $$

This result can also be interpreted in the following way. Consider an insured who is alive at time t. If he/she survives to time $t + 1$, the insurance company has to set aside an amount of ${}_{t+1}V$ at that time. If death occurs between times t and $t + 1$, the company needs an amount of S_{t+1} at time $t + 1$ to pay the death benefit. Since S_{t+1} differs from ${}_{t+1}V$ by $S_{t+1} - {}_{t+1}V$, we call $S_{t+1} - {}_{t+1}V$ the death strain at risk. The probability that the amount of the death strain at risk will actually be needed is q_{x+t}.

Therefore, the expected value of the amount needed in addition to ${}_{t+1}V$ at time $t+1$ is

$$p_{x+t}\,0 + q_{x+t}(S_{t+1} - {}_{t+1}V) = q_{x+t}(S_{t+1} - {}_{t+1}V).$$

The expression $q_{x+t}(S_{t+1} - {}_{t+1}V)$ is referred to as the expected death strain or the cost of insurance. Summing up the expected death strains over the n_t policies in force at time t, we get the total expected death strain (*TEDS*):

$$TEDS = \sum_{k=1}^{n_t} q_{x+t}(S_{t+1} - {}_{t+1}V) = n_t\, q_{x+t}(S_{t+1} - {}_{t+1}V). \tag{11}$$

On the other hand, the amount actually needed in addition to $\sum_{k=1}^{n_t} {}_{t+1}V$ at time $t+1$ is called the total actual death strain (*TADS*):

$$TADS = \sum_{k=1}^{n_t - n_{t+1}} (S_{t+1} - {}_{t+1}V) = (n_t - n_{t+1})\,(S_{t+1} - {}_{t+1}V). \tag{12}$$

So from (10), (11), and (12) we get the following equation

Total mortality profit for a year
= Total expected death strain - Total actual death strain. (13)

Note that the death strain at risk can be negative. For example, an annuity insurance does not contain any death benefits, so the death strain is a negative number: $-{}_{t+1}V$. This is quite reasonable, considering that in this situation the death of the policyholder is a "gain" for the insurance company, since no more benefits have to be paid on the policy afterwards.

Since

$$TEDS - TADS = (S_{t+1} - {}_{t+1}V)\,(n_t\, q_{x+t} - (n_t - n_{t+1})),$$

we can check very easily whether there is a mortality profit or loss on a group of policies for the year $t+1$. Note that $n_t\, q_{x+t}$ is the expected number of people dying between times t and $t+1$ from a group of n_t people alive at time t, and $n_t - n_{t+1}$ is the actual number of people dying during this period. So if the death strain at risk; that is, $S_{t+1} - {}_{t+1}V$ is positive, and

a) fewer people die between t and $t+1$ $(n_t - n_{t+1})$ than expected $(n_t\, q_{x+t})$, there is a mortality profit,

b) more people die between t and $t+1$ than expected, there is a mortality loss.

On the other hand, if the death strain at risk is negative, and

a) fewer people die between t and $t+1$ than expected, there is a mortality loss,

b) more people die between t and $t+1$ than expected, there is a mortality profit.

EXAMPLE 2.1. An insurance company issues 20 year endowment insurances of \$2000 to lives aged 50 on January 1, 1980. The premiums are payable annually. On January 1, 1993, there are still 800 policyholders alive. During the year of 1993, 13 of them die. Calculate the mortality profit or loss for the year 1993. Use a 6% annual interest rate.

Solution: First of all, we have to determine the annual premium for a policy. It is

$$\begin{aligned}
P &= 2000\frac{A_{50:\overline{20}|}}{\ddot{a}_{50:\overline{20}|}} \\
&= 2000\frac{M_{50} - M_{70} + D_{70}}{N_{50} - N_{70}} \\
&= 2000\,\frac{1210.1957 - 576.7113 + 1119.94}{64467.45 - 9597.05} \\
&= \$63.91.
\end{aligned}$$

If we want to use (5), we have to know the reserves at the beginning and at the end of 1993.

On January 1, 1993, the reserve for a policy in force is

$$\begin{aligned}
{}_{13}V &= 2000\,A_{63:\overline{7}|} - P\,\ddot{a}_{63:\overline{7}|} \\
&= 2000\left(1 - \frac{\ddot{a}_{63:\overline{7}|}}{\ddot{a}_{50:\overline{20}|}}\right) \\
&= 2000\left(1 - \frac{N_{63} - N_{70}}{D_{63}}\cdot\frac{D_{50}}{N_{50} - N_{70}}\right) \\
&= 2000\left(1 - \frac{20726.94 - 9597.05}{1991.37}\cdot\frac{4859.30}{64467.45 - 9597.05}\right) \\
&= \$1010.07.
\end{aligned}$$

The easiest way to obtain the reserve one year later is to use the recursive relationship (17) of Section 5.1:

$${}_{14}V = ({}_{13}V + P)\,u_{63} - 2000\,k_{63}$$

$$= \frac{(1010.07 + 63.91)\,D_{63} - 2000\,C_{63}}{D_{64}}$$

$$= \frac{1073.98 \times 1991.37 - 2000 \times 33.5925}{1845.06}$$
$$= \$1122.73.$$

Moreover,

$$B_{13} = 0,$$

$$P_{13} = P = 63.91,$$

$$S_{14} = 2000,$$

$$n_{13} = 800,$$

$$n_{14} = 800 - 13 = 787$$

and

$$i = 0.06.$$

Now using (5), the mortality profit is

$$\begin{aligned} n_{13}\,({}_{13}V - B_{13} + P_{13})(1 + i) - n_{14}\,{}_{14}V - (n_{13} - n_{14})S_{14} \\ = 800(1010.07 + 63.91)\ 1.06 - 787 \times 1122.73 - 13 \cdot 2000 \\ = \$1146.53. \end{aligned}$$

That means, there is a mortality profit of \$1146.53 for the year 1993.
Alternatively, we can use (13). Since

$${}_{14}V = 1122.73,$$

$$S_{14} = 2000,$$

$$n_{13} = 800,$$

$$n_{14} = 787,$$

and

$$q_{63} = 0.0178812,$$

we find that the death strain at risk is

$$S_{14} - {}_{14}V = 2000 - 1122.73 = 877.27.$$

Therefore,

$$\begin{aligned} TEDS &= n_{13}\, q_{63}(S_{14} - {}_{14}V) \\ &= 800 \times 0.0178812 \times 877.27 \\ &= 12549.31 \end{aligned}$$

and

$$\begin{aligned} TADS &= (n_{13} - n_{14})(S_{14} - {}_{14}V) \\ &= 13 \times 877.27 \\ &= 11404.51. \end{aligned}$$

Thus, the mortality profit is

$$\begin{aligned} TEDS - TADS &= 12549.31 - 11404.51 \\ &= \$1144.80. \end{aligned}$$

Therefore, we get a mortality profit of $\$1144.80$ for the year 1993.

The difference between $\$1144.80$ and the result of the first method, $\$1146.53$ is due to round-off errors.

Note that even without detailed computations, we can see that the mortality profit is positive. Indeed, this must be true since the death strain at risk, 877.27 is positive and the expected number of policyholders dying in 1993, $n_{13} \cdot q_{63} = 800 \times 0.0178812 = 14.30$ is greater than the actual number of deaths, which is 13.

The following examples show that the mortality profit on a group of policies can often be determined without knowing the number of policies. Of course, then we must have some other information. For example, in the case of a temporary or whole life or an endowment insurance, this information can be the total sum insured for the policies in force at the beginning and at the end of the year. In the case of a life annuity, the total annual amount of the annuities in force is the extra information we need.

EXAMPLE 2.2. An insurance company issues whole life insurances to lives aged 40 on January 1, 1990. The insurances are purchased by annual premiums. On January 1, 1992, the total sum insured is $\$5{,}024{,}000$, and in 1992, $\$20{,}000$ is paid out in death benefits. Using a 6% annual rate of interest, determine the mortality profit or loss for the year 1992.

Solution: Let us denote the death benefit of a policy by S. Then the annual premium for the insurance can be expressed as

$$P = S\frac{A_{40}}{\ddot{a}_{40}} = S\frac{0.1613242}{14.81661} = S \times 0.0108881.$$

On January 1, 1992, the reserve for a policy in force is

$${}_2V = S\,A_{42} - P\,\ddot{a}_{42}$$

$$
\begin{aligned}
&= S\left(1 - \frac{\ddot{a}_{42}}{\ddot{a}_{40}}\right) \\
&= S\left(1 - \frac{14.55102}{14.81661}\right) \\
&= S \times 0.0179252.
\end{aligned}
$$

Furthermore, the reserve for a policy in force one year later is

$$
\begin{aligned}
{}_3V &= S\left(1 - \frac{\ddot{a}_{43}}{\ddot{a}_{40}}\right) \\
&= S\left(1 - \frac{14.41022}{14.81661}\right) \\
&= S \times 0.0274280.
\end{aligned}
$$

Thus using (5), the mortality profit is

$$
\begin{aligned}
&n_2\,({}_2V + P_2)(1 + i) - n_3\,{}_3V - (n_2 - n_3)\,S_3 \\
&\quad = n_2(S \times 0.0179252 + S \times 0.0108881)1.06 - n_3\,S \times 0.0274280 - (n_2 - n_3)\,S \\
&\quad = n_2 S \times 0.0305421 - n_3\,S \times 0.0274280 - (n_2 - n_3)\,S.
\end{aligned}
$$

Note that $n_2\,S$ is the total sum insured at the beginning of 1992, therefore

$$n_2\,S = 5024000.$$

On the other hand, $(n_2 - n_3)\,S$ is the total death benefit payment in 1992, hence

$$(n_2 - n_3)\,S = 20000.$$

Thus, we obtain

$$n_3\,S = n_2\,S - (n_2 - n_3)\,S = 5024000 - 20000 = 5004000.$$

Therefore, the mortality profit is

$$5024000 \times 0.0305421 - 5004000 \times 0.0274280 - 20000 = -3806.20.$$

So there is a mortality loss of \$3806.20 for the year 1992.

Alternatively, we can use (13). The death strain at risk is

$$S_3 - {}_3V = S - S \times 0.0274280 = S \times 0.972572$$

thus

$$TEDS = n_2\,q_{42}\,(S_3 - {}_3V)$$

$$\begin{aligned} &= 0.0032017\, n_2\, S \times 0.972572 \\ &= 0.0032017 \times 5024000 \times 0.972572 \\ &= 15644.15 \end{aligned}$$

and

$$\begin{aligned} TADS = (n_2 - n_3)(S - {}_3V) &= (n_2 - n_3)\, S \times 0.972572 \\ &= 20000 \times 0.972572 \\ &= 19451.44. \end{aligned}$$

Thus, the mortality profit is

$$\begin{aligned} TEDS - TADS &= 15644.15 - 19451.44 \\ &= -3807.29. \end{aligned}$$

So we get a mortality loss of $3807.29. The difference between $3806.20 and $3807.29 is due to round-off errors.

EXAMPLE 2.3. An insurance company issues 10 year annuities-due deferred 5 years to lives aged 65 on January 1, 1985, when a single premium is payable. Using a 6% annual rate of interest find the mortality profit or loss

a) for 1986, if the total annual amount of the annuities is $160,000 for the policies in force at the beginning of the year and $156,000 for the policies in force at the end of the year.

b) for 1991, if the total annual amount of the annuities is $137,600 for the policies in force at the beginning of the year and $132,800 for the policies in force at the end of the year.

Solution: Let us denote the survival benefit payable annually on a policy by B. Since premium is only paid in the year 1985, we do not need it in the computation of the mortality profit for the years 1986 and 1991.

a) At the beginning of 1986, the reserve for a policy in force is

$$\begin{aligned} {}_1V &= B\; {}_{4|}\ddot{a}_{66:\overline{10|}} \\ &= B\frac{N_{70} - N_{80}}{D_{66}} \\ &= B\frac{9597.05 - 2184.81}{1575.71} \\ &= B \times 4.7040636, \end{aligned}$$

and the reserve at the end of 1986 is

$$\begin{aligned} {}_2V &= B\; {}_{3|}\ddot{a}_{67:\overline{10|}} \\ &= B\frac{N_{70} - N_{80}}{D_{67}} \end{aligned}$$

$$= B\frac{9597.05 - 2184.81}{1451.90}$$
$$= B \times 5.1052001.$$

Since no benefit payments can take place in the year 1986, using (5), the mortality profit is

$$n_1\, {}_1V(1 + i) - n_2\, {}_2V = n_1\, B \times 4.7040636 \times 1.06 - n_2\, B \times 5.1052001$$
$$= n_1\, B \times 4.9863074 - n_2\, B \times 5.1052001.$$

Now, n_1B is the total annual amount of the annuities for the policies in force at the beginning of 1986, so

$$n_1\, B = 160{,}000.$$

Furthermore, n_2B is the total annual amount of the annuities for the policies in force at the end of 1986, thus

$$n_2\, B = 156{,}000.$$

Therefore, the mortality profit is

$$160{,}000 \times 4.9863074 - 156{,}000 \times 5.1052001 = \$1397.97$$

for the year 1986.

Let us see what result we get if we use (13). The death strain at risk is

$$S_2 - {}_2V = -B \times 5.1052001$$

so

$$\begin{aligned} TEDS &= n_1\, q_{66}\, (-{}_2V) \\ &= n_1 \times 0.0232871\, (-B \times 5.1052001) \\ &= -n_1\, B \times 0.1188853 \\ &= -160{,}000 \times 0.1188853 \\ &= -19021.65 \end{aligned}$$

and

$$\begin{aligned} TADS &= (n_1 - n_2)(-{}_2V) \\ &= (n_1 - n_2)(-B \times 5.1052001) \\ &= -(n_1B - n_2B)\ 5.1052001 \\ &= -(160{,}000 - 156{,}000)\ 5.1052001 \\ &= -20420.80. \end{aligned}$$

Thus the mortality profit is

$$TEDS - TADS = -19021.65 - (-20420.80) = \$1399.15.$$

The difference between \$1397.97 and \$1399.15 is due to round-off errors.

b) At the beginning of 1991, the reserve for a policy in force is

$$\begin{aligned}
{}_6V &= B\,\ddot{a}_{71:\overline{9}|} \\
&= B\frac{N_{71} - N_{80}}{D_{71}} \\
&= B\frac{8477.11 - 2184.81}{1021.49} \\
&= B \times 6.1599232,
\end{aligned}$$

and the reserve at the end of 1991 is

$${}_7V = B\,\ddot{a}_{72:\overline{8}|} = B\frac{N_{72} - N_{80}}{D_{72}} = B\frac{7455.62 - 2184.81}{928.72} = B \times 5.6753489.$$

There is a survival benefit payable at the beginning of 1991, thus from (5), the mortality profit is

$$\begin{aligned}
n_6({}_6V - B)(1 + i) - n_7\,{}_7V &= n_6(B \times 6.1599232 - B)1.06 - n_7\,B \times 5.6753489 \\
&= n_6\,B \times 5.1599232 \times 1.06 - n_7\,B \times 5.6753489 \\
&= 137600 \times 5.1599232 \times 1.06 - 132800 \times 5.6753489 \\
&= -1080.58.
\end{aligned}$$

Thus there is a mortality loss of \$1080.58 for the year 1991.

If we want to apply (13), we have to determine the death strain at risk first. It is

$$S_7 - {}_7V = -B \times 5.6753489.$$

Hence

$$\begin{aligned}
TEDS &= n_6\,q_{71}\,(-{}_7V) \\
&= n_6\,0.0362608\,(-B \times 5.6753489) \\
&= -n_6\,B \times 0.2057927 \\
&= -137600 \times 0.2057927 \\
&= -28317.08
\end{aligned}$$

and

$$\begin{aligned}
TADS &= (n_6 - n_7)\,(-{}_7V) \\
&= (n_6 - n_7)(-B \times 5.6753489) \\
&=- (n_6B - n_7B)\,5.6753489 \\
&= -\,(137600 - 132800)\,5.6753489 \\
&= -27241.67.
\end{aligned}$$

Thus the mortality profit is

$$TEDS - TADS = -28317.08 - (-27241.67) = -1075.41.$$

So there is a mortality loss of \$1075.41 for the year 1991. The difference between \$1080.58 and \$1075.41 is due to round-off errors.

In Section 5.1, we studied three types of insurances which make benefit payments beyond the end of the year of death: guaranteed annuities, family income benefits, and term certain insurances. The reader is advised at this point to review this part of Section 5.1. We showed that the prolonged benefit payments of these insurances is equivalent to a one time death benefit payment at the end of the year of death whose amount is the present value of the future payments. If we want to find the mortality profit for these insurances, we have to make these substitutions first.

EXAMPLE 2.4. An insurance company issues 20 year family income benefits of \$3000 per annum to lives aged 45 on January 1, 1985 whose premiums are payable annually for 15 years. At the beginning of 1990, 670 of the insured are still alive, from whom 5 die during the year. Based on a 6% annual rate of interest, find the mortality profit on the group for the year 1990.

Solution: The annual premium for the insurance is

$$P = 3000\frac{a_{\overline{20}|} - a_{45:\overline{20}|}}{\ddot{a}_{45:\overline{15}|}}$$

$$= 3000\frac{a_{\overline{20}|} - \dfrac{N_{46} - N_{66}}{D_{45}}}{\dfrac{N_{45} - N_{60}}{D_{45}}}$$

$$= 3000\frac{11.4699 - \dfrac{87296.23 - 15183.86}{6657.69}}{\dfrac{93953.92 - 27664.55}{6657.69}}$$

$$= \$192.37.$$

On January 1, 1990, if the insured is alive, the reserve for the policy is

$$_5V = 3000\,(a_{\overline{15}|} - a_{50:\overline{15}|}) - P\,\ddot{a}_{50:\overline{10}|}$$

$$= 3000\left(a_{\overline{15}|} - \frac{N_{51} - N_{66}}{D_{50}}\right) - 192.37\frac{N_{50} - N_{60}}{D_{50}}$$

$$= 3000\left(9.7122 - \frac{59608.16 - 15183.86}{4859.30}\right) - 192.37\frac{64467.45 - 27664.55}{4859.30}$$
$$= \$253.29$$

and the reserve one year later is

$$\begin{aligned}
{}_6V &= 3000\,(a_{\overline{14}|} - a_{51:\overline{14}|}) - P\,\ddot{a}_{51:\overline{9}|} \\
&= 3000\left(a_{\overline{14}|} - \frac{N_{52} - N_{66}}{D_{51}}\right) - P\frac{N_{51} - N_{60}}{D_{51}} \\
&= 3000\left(9.2950 - \frac{55051.05 - 15183.86}{4557.10}\right) - 192.37\frac{59608.16 - 27664.55}{4557.10} \\
&= \$291.45.
\end{aligned}$$

We also have

$$B_5 = 0,$$

$$P_5 = P = 192.37,$$

$$\begin{aligned}
S_6 = 3000\,\ddot{a}_{\overline{20-6+1}|} &= 3000\,\ddot{a}_{\overline{15}|} \\
&= 3000(1 + a_{\overline{14}|}) \\
&= 3000(1 + 9.2950) \\
&= 3000 \times 10.2950 \\
&= 30885
\end{aligned}$$

$$n_5 = 670$$

and

$$n_6 = 670 - 5 = 665.$$

Thus using (5), the mortality profit is

$$\begin{aligned}
n_5({}_5V - B_5 + P_5)(1 + i) - n_6\,{}_6V - (n_5 - n_6)\,S_6 \\
&= 670(253.29 + 192.37)\,1.06 - 665 \times 291.45 - 5 \times 30885 \\
&= -31731.52.
\end{aligned}$$

Hence there is a mortality loss of \$31731.52 for the year 1990.

If we want to use (13), first we have to determine the death strain at risk. It is

$$S_6 - {}_6V = 30885 - 291.45 = 30593.55.$$

Therefore we have

$$
\begin{aligned}
TEDS &= n_5\, q_{50}(S_6 - {}_6V) \\
&= 670 \times 0.0059199 \times 30593.55 \\
&= 121344.21
\end{aligned}
$$

and

$$
\begin{aligned}
TADS &= (n_5 - n_6)(S_6 - {}_6V) \\
&= 5 \times 30593.55 \\
&= 152967.75.
\end{aligned}
$$

Since

$$TEDS - TADS = 121344.21 - 152967.75 = -31623.54,$$

we find a mortality loss of $31623.54 for the year 1990. The difference between $31731.52 and $31623.54 is due to round-off errors.

Since the death strain at risk is positive and the expected number of deaths in 1992, $n_5\, q_{50} = 670 \times 0.0059199 = 3.97$ is less than 5, the actual number of deaths, it follows immediately that there must be a mortality loss for the year, as the detailed computation has shown.

PROBLEMS

2.1. An insurance company issues 25 year term insurances of $12,000 to lives aged 30 on January 1, 1985. The premiums for the insurances are payable annually. On January 1, 1990, the number of policyholders still alive is 540 from whom 2 die during the year. Based on a 6% annual rate of interest, find the mortality profit or loss for the year 1990.

2.2. An insurance company issues 15 year annuities-due of $2000 per annum deferred 4 years to lives aged 40 on January 1, 1980. The premium is payable yearly during the 4 years of the deferment period. If the number of policies in force is 950 on January 1, 1982, 945 on January 1, 1983, 942 on January 1, 1984, and 940 on January 1, 1985, find the mortality profit or loss

a) for the year 1982.
b) for the year 1983.
c) for the year 1984.

Base the computations on a 6% annual rate of interest.

2.3. An insurance company issues 10 year endowment insurances to lives aged 45 on January 1, 1990. The premiums are payable annually. On

January 1, 1992, the total sum at risk is $2,400,000. During the year of 1992, the total death benefit payment is $8000. Based on a 6% annual interest rate, determine the mortality profit or loss for the year 1992.

2.4. An insurance company issues 20 year annuities-due to lives aged 55 on January 1, 1982 when the single premiums are payable. The amount of the total annuity payments if $1,460,000 in 1990 dropping to $1,430,000 in 1991. Using a 6% annual rate of interest, obtain the mortality profit or loss for the year 1990.

2.5. An insurance company issues 15 year term certain insurances of $7000 to lives aged 40 on January 1, 1980. The premiums are payable annually. From the 800 policyholders alive at the beginning of 1990, 796 survive to the end of the year. Find the mortality profit or loss for the year 1990 based on a 6% annual interest rate.

2.6. An insurance company issues 15 year annuity-immediates of $1500 per annum with a guaranteed payment period of 10 years to lives aged 50 on January 1, 1990 when the single premiums are payable. Determine the mortality profit or loss for the first year of the insurance if from the 830 insured who have bought the insurance, 4 die within one year. Use a 6% annual rate of interest.

5.3. Modified Reserves

If the reserves are based on gross (office) premiums, they are called modified reserves. In order to distinguish the modified reserve from the net premium reserve more clearly, we can use the notation V^{net} for the net premium reserve and V^{mod} for the modified reserve.

We will study the modified reserves in the context of the special expense structure given in Theorem 2.1 of Section 4.2. This model contains premiums payable in the form of an n year level annuity-due (n is an integer or infinity). Each time the premium is paid, a renewal expense k times the annual premium and other renewal expenses of c are incurred. In addition to that, there is an initial expense of I.

In Theorem 2.1 of Section 4.2 we showed that the gross premium P'' can be obtained as

$$P'' = \frac{1}{1-k}\left(P + c + \frac{I}{\ddot{a}_{x:\overline{n}|}}\right) \qquad (1)$$

where P is the net premium.

Let us determine the modified reserve at time t ($t = 0,1,\dots,n$). If $t = 0$ then the initial expense I is considered to have already been incurred so it is taken into account in the retrospective reserve. Thus

$$ {}_0V^{mod} = -I \tag{2}$$

which is always negative if the initial expense is greater than zero. There is also another way of looking at the modified reserve at time zero. We can argue that unless the first premium is paid, the policy cannot be considered as being in force. So it is reasonable to determine the reserve just after the first premium payment. Then P'' has already been paid, but a part of it: $kP'' + c$ is used to cover renewal expenses. What is left is

$$P'' - kP'' - c = (1 - k)P'' - c = (1 - k)\frac{1}{1 - k}\left(P + c + \frac{I}{\ddot{a}_{x:\overline{n}|}}\right) - c = P + \frac{I}{\ddot{a}_{x:\overline{n}|}}.$$

Subtracting the initial expense from this, we get

$$P + \frac{I}{\ddot{a}_{x:\overline{n}|}} - I = P - I\left(1 - \frac{1}{\ddot{a}_{x:\overline{n}|}}\right). \tag{3}$$

Formula (3) often gives a better picture of the situation at the commencement of the insurance than (2).

If $t = 1,2,\dots,n$, then ${}_tV^{mod}$ equals the expected present value of future benefits plus the expected present value of future expenses minus the expected present value of future gross premiums. On the other hand, ${}_tV^{net}$ equals the expected present value of future benefits minus the expected present value of future net premiums. Thus

$${}_tV^{mod} = EPV_t(\text{cash flow of benefits after } t) + (kP'' + c)\ddot{a}_{x+t:\overline{n-t}|} - P''\ddot{a}_{x+t:\overline{n-t}|} \tag{4}$$

and

$${}_tV^{net} = EPV_t(\text{cash flow of benefits after } t) - P\ddot{a}_{x+t:\overline{n-t}|}. \tag{5}$$

Subtracting (5) from (4), we obtain

$${}_tV^{mod} - {}_tV^{net} = (kP'' + c)\ddot{a}_{x+t:\overline{n-t}|} - P''\ddot{a}_{x+t:\overline{n-t}|} + P\ddot{a}_{x+t:\overline{n-t}|}$$

that is

$${}_tV^{mod} - {}_tV^{net} = ((k - 1)P'' + c + P)\ddot{a}_{x+t:\overline{n-t}|}.$$

So using (1) we get

$$\begin{aligned} {}_tV^{mod} - {}_tV^{net} &= \left((k-1)\frac{1}{1-k}\left(P + c + \frac{I}{\ddot{a}_{x:\overline{n}|}} \right) + c + P \right) \ddot{a}_{x+t:\overline{n-t}|} \\ &= \left(-P - c - \frac{I}{\ddot{a}_{x:\overline{n}|}} + c + P \right) \ddot{a}_{x+t:\overline{n-t}|} \\ &= -I\frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}}. \end{aligned}$$

Therefore,

$${}_tV^{mod} = {}_tV^{net} - I\frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}} \text{ for } t = 1,2,\dots,n. \tag{6}$$

If n is infinity, (6) takes on the form

$${}_tV^{mod} = {}_tV^{net} - I\frac{\ddot{a}_{x+t}}{\ddot{a}_x}, \text{ for } t = 1,2,\dots . \tag{7}$$

If $t \geq n$, then the future cash flow consists of benefit payments only, it does not contain premium payments and expenses any more. Therefore, using prospective reserves, we get

$${}_tV^{mod} = {}_tV^{net}, \text{ for } t \geq n. \tag{8}$$

Note that if $t = n$ then (6) and (8) give the same result since

$$\frac{\ddot{a}_{x+n:\overline{n-n}|}}{\ddot{a}_{x:\overline{n}|}} = \frac{\ddot{a}_{x+n:\overline{0}|}}{\ddot{a}_{x:\overline{n}|}} = 0.$$

In certain cases, the modified reserve can be expressed as a simple function of the net premium reserve and the initial expense.

For example, in Section 5.1, we showed that the net premium reserve for an n year endowment insurance is

$${}_tV^{net}_{x:\overline{n}|} = 1 - \frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}}.$$

So in this case, (6) can be rewritten as

$${}_tV^{mod}_{x:\overline{n}|} = {}_tV^{net}_{x:\overline{n}|} - I\,(1 - {}_tV^{net}_{x:\overline{n}|}).$$

A similar expression can be obtained for a whole life insurance. Since the net premium reserve is

$$ {}_tV_x^{net} = 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}, $$

(7) implies

$$ {}_tV_x^{mod} = {}_tV_x^{net} - I(1 - {}_tV_x^{net}). $$

Note that the expressions (2), (3), (6), and (7) do not contain c and k. That means, the modified reserve does not depend on the renewal expenses. This is easily understandable if we recall that after Theorem 2.1 of Section 4.2 we saw that each premium covers the renewal expenses of the respective year completely. Thus, it is not necessary to make reserves for these expenses, so if we want to determine the modified reserve it is enough to consider the initial expenses. Reserves that take into account the initial expenses are called Zillmerized reserves named after the 19th century German actuary August Zillmer.

For the rest of this section, we will ignore renewal expenses; that is, we assume $c = 0$ and $k = 0$. We will focus on the effect of the initial expenses on the reserves. The gross annual premium in this case is

$$ P'' = P + \frac{I}{\ddot{a}_{x:\overline{n}|}}. \tag{9} $$

In other words, we have a series of premium payments

$$ P_t'' = P_t + \frac{I}{\ddot{a}_{x:\overline{n}|}}, \ t = 0,1,\ldots,n-1 \tag{10} $$

where $P_t = P$ is the level annual net premium and P_t'' denotes the gross premium due at time t. So each gross premium can be split into two parts. One part, P is needed to pay the benefits while the other part, $\frac{I}{\ddot{a}_{x:\overline{n}|}}$, is used to recover the initial expense I over the time. Thus it takes rather long ($n - 1$ years), before the initial expense can finally be settled. However, since the initial expense is incurred right at the beginning of the policy, the insurance company usually prefers to settle it as soon as possible. This may be achieved by dividing the gross premiums in a different way. Let us assume that P_t'' is split as

$$ P_t'' = P_t^* + I_t^*, \ t = 0,1,\ldots,n-1 \tag{11} $$

where

$$ 0 \le P_t^* \le P_t'', \ t = 0,1,\ldots,n-1 \tag{12} $$

and

$$\sum_{t=0}^{n-1} {}_tE_x\, P_t^* = \sum_{t=0}^{n-1} {}_tE_x\, P_t. \tag{13}$$

Then the expected present value at time $t = 0$ of the cash flow consisting of $P_0^*, P_1^*, \ldots, P_{n-1}^*$ equals the expected present value at time $t = 0$ of the benefit payments. Therefore, P_t^*'s can be used as modified net premiums. Moreover, it follows from (10), (11), and (13) that

$$\begin{aligned}
\sum_{t=0}^{n-1} {}_tE_x\, I_t^* &= \sum_{t=0}^{n-1} {}_tE_x\, (P_t'' - P_t^*) \\
&= \sum_{t=0}^{n-1} {}_tE_x\, P_t'' - \sum_{t=0}^{n-1} {}_tE_x\, P_t^* \\
&= \sum_{t=0}^{n-1} {}_tE_x\, P_t'' - \sum_{t=0}^{n-1} {}_tE_x\, P_t \\
&= \sum_{t=0}^{n-1} {}_tE_x\, (P_t'' - P_t) \\
&= \sum_{t=0}^{n-1} {}_tE_x\, \frac{I}{\ddot{a}_{x:\overline{n}|}} \\
&= I\frac{\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} \\
&= I.
\end{aligned}$$

Thus the initial expense I can be recovered from the cash flow $I_0^*, I_1^*, \ldots, I_{n-1}^*$.

If we base the net premium reserve calculation on the modified net premiums, we obtain the modified net premium reserves ${}_tV^{*net}$. Then

$$\begin{aligned}
{}_tV^{*net} = {} & EPV_t(\text{cash flow of benefits after } t) \\
& - EPV_t(\text{cash flow of modified net premiums after } t).
\end{aligned}$$

The modified net premium reserve can also be denoted by ${}_tV^{mod}$. However, for the sake of clarity, we will use this notation only for the modified reserve defined by (4).

When we choose the modified net premiums $P_0^*, P_1^*, \ldots, P_{n-1}^*$ we want to make sure they will not give rise to negative reserves:

$${}_tV^{*net} \geq 0, \; t = 0,1,2,\ldots,n.$$

There are various ways of determining modified net premiums.

One possibility is to take off the whole amount of the initial expense from the first gross premium P_0''. This way we can settle the initial expense immediately at the start of the policy, so we do not have to worry about it afterwards. The following theorem shows that this can always be done if the modified reserves are nonnegative.

THEOREM 3.1. *Consider an insurance with an initial expense of I whose gross premiums are payable in the form of an n year level annuity-due (n is a positive integer or infinity). Let P'' denote the level gross annual premium. Assume*

$$_tV^{mod} \geq 0, \quad t = 1,2,\ldots,n-1. \tag{14}$$

Let us define the modified net premiums as

$$P_0^* = P'' - I, \tag{15}$$

and

$$P_t^* = P'', \quad t = 1,2,\ldots,n-1. \tag{16}$$

Then we get

$$_0V^{*net} = 0 \tag{17}$$

and

$$_tV^{*net} = {_tV^{mod}} \geq 0, \quad t \geq 1. \tag{18}$$

Proof: Let the level net premium be P per annum. From (9), (15), and (16) we get

$$\begin{aligned}\sum_{t=0}^{n-1} {_tE_x}\, P_t^* &= P'' - I + \sum_{t=1}^{n-1} {_tE_x}\, P'' \\ &= \sum_{t=0}^{n-1} {_tE_x}\, P'' - I \\ &= \sum_{t=0}^{n-1} {_tE_x}\left(P + \frac{I}{\ddot{a}_{x:\overline{n}|}}\right) - I \\ &= \sum_{t=0}^{n-1} {_tE_x}\, P + I\frac{\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} - I\end{aligned}$$

$$= \sum_{t=0}^{n-1} {}_tE_x\, P.$$

Thus (13) is satisfied.

Equating the retrospective modified reserve at time zero

$${}_0V^{mod} = -I$$

to the prospective modified reserve

$$\begin{aligned} {}_0V^{mod} &= EPV_0 \text{ (cash flow of benefits between times } t = 0 \text{ and } t = 1) \\ &\quad + {}_1E_x\, {}_1V^{mod} - P'' \end{aligned}$$

we get

$$\begin{aligned} P'' - I &= EPV_0 \text{ (cash flow of benefits between times } t = 0 \text{ and } t = 1) \\ &\quad + {}_1E_x\, {}_1V^{mod}. \end{aligned} \tag{19}$$

Now, the first term on the right hand side of (19) is nonnegative. Since ${}_1V^{mod} \geq 0$, the second term is also nonnegative. Thus

$$0 \leq P'' - I$$

and therefore

$$0 \leq P_0^* \leq P''.$$

Moreover

$$0 \leq P_t^* \leq P'', \quad t = 1,2,\ldots,n-1$$

is true because of (16). Hence (12) is also satisfied. Since any net premium reserve is zero at time $t_0 = 0$, (17) is true.

If $t \geq 1$, then

$$\begin{aligned} {}_tV^{*net} &= EPV_t \text{ (cash flow of benefits after } t) \\ &\quad - EPV_t \text{ (cash flow of modified net premiums after } t). \end{aligned}$$

However, it follows from (16) that the modified net premiums after t coincide with the gross premiums. Thus

$$\begin{aligned} {}_tV^{*net} &= EPV_t \text{ (cash flow of benefits after } t) \\ &\quad - EPV_t \text{ (cash flow of gross premiums after } t) = {}_tV^{mod}. \end{aligned} \tag{20}$$

Hence (18) follows from (14) and (20), if $1 \le t \le n - 1$. If $t \ge n$, then there are no premium payments after time t, hence

$$_tV^{*net} = {}_tV^{mod} = EPV_t \text{ (cash flow of benefits after } t) \ge 0$$

which proves (18) for $t \ge n$. ■

Next we show that if the cash flow of the benefits of an insurance satisfy the conditions of Theorem 1.1 of Section 5.1 and the premiums are payable in the form of an n year level annuity-due then $_1V^{mod} \ge 0$ implies $_tV^{mod} \ge 0$, $t = 2,3,\ldots,n - 1$. Thus in this case, it is enough to check the nonnegativity of the modified reserve at duration $t = 1$.

Indeed, we have seen after Theorem 1.1 of Section 5.1 that the level annual premium satisfies the conditions of Theorem 1.1 of Section 5.1, thus using the theorem we find that

$$r(t) = \frac{_tV^{net}}{P\,\ddot{a}_{x+t:\overline{n-t}|}}, \quad t = 1,2,\ldots,n - 1$$

is increasing in t.

Now from (6) we get

$$\begin{aligned} _tV^{mod} &= {}_tV^{net} - I\frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}} \\ &= \ddot{a}_{x+t:\overline{n-t}|}\left(\frac{_tV^{net}}{\ddot{a}_{x+t:\overline{n-t}|}} - \frac{I}{\ddot{a}_{x:\overline{n}|}}\right), \quad t = 1,2,\ldots,n - 1. \end{aligned}$$

Therefore,

$$_tV^{mod} \ge 0$$

is equivalent to

$$\frac{_tV^{net}}{\ddot{a}_{x+t:\overline{n-t}|}} \ge \frac{I}{\ddot{a}_{x:\overline{n}|}}, \quad t = 1,2,\ldots,n - 1.$$

Since $r(t)$ is increasing in t, $\dfrac{_tV^{net}}{\ddot{a}_{x+t:\overline{n-t}|}}$ is also increasing in t. Therefore, if

$$_1V^{mod} \ge 0$$

then

$$\frac{_1V^{net}}{\ddot{a}_{x+1:\overline{n-1}|}} \ge \frac{I}{\ddot{a}_{x:\overline{n}|}}.$$

Hence

$$\frac{{}_tV^{net}}{\ddot{a}_{x+t:\overline{n-t|}}} \geq \frac{I}{\ddot{a}_{x:\overline{n|}}}, \quad t = 1,2,3,\ldots,n-1$$

and

$${}_tV^{mod} \geq 0, \quad t = 1,2,3,\ldots,n-1,$$

which proves our statement.

Using (6), the condition

$${}_1V^{mod} \geq 0 \tag{21}$$

can also be expressed as

$$I \leq {}_1V^{net} \frac{\ddot{a}_{x:\overline{n|}}}{\ddot{a}_{x+1:\overline{n-1|}}}. \tag{22}$$

Furthermore, from (9) and (19) we get

$$\begin{aligned}
&{}_1E_x\,{}_1V^{mod} \\
&= P'' - I - EPV_0\,(\text{cash flow of benefits between times } t{=}0 \text{ and } t{=}1) \\
&= P - I\left(1 - \frac{1}{\ddot{a}_{x:\overline{n|}}}\right) - EPV_0\,(\text{cash flow of benefits between times } t{=}0 \text{ and } t{=}1)\,.
\end{aligned}$$

Thus (21) is equivalent to

$$I \leq \frac{P - EPV_0(\text{cash flow of benefits between times } t{=}0 \text{ and } t{=}1)}{1 - \dfrac{1}{\ddot{a}_{x:\overline{n|}}}}\,. \tag{23}$$

The largest I for which (21) is still true is called the Zillmer maximum, denoted by $I_{Zillmer}$. So if the initial expense is the Zillmer maximum, we get

$${}_1V^{mod} = 0. \tag{24}$$

Using (22) or (23) we obtain the following two expressions for the Zillmer maximum:

$$I_{Zillmer} = {}_1V^{net} \frac{\ddot{a}_{x:\overline{n|}}}{\ddot{a}_{x+1:\overline{n-1|}}} \tag{25}$$

and

$$I_{Zillmer} = \frac{P - EPV_0(\text{cash flow of benefits between times } t{=}0 \text{ and } t{=}1)}{1 - \dfrac{1}{\ddot{a}_{x:\overline{n}|}}}. \tag{26}$$

It follows from (15), (19), and (24) that using the Zillmer maximum as the initial expense, we have

$$P_0^* = EPV_0(\text{cash flow of benefits between times } t{=}0 \text{ and } t = 1). \tag{27}$$

That means, the modified net premium at time zero is exactly the amount of a single premium charged for providing the benefit payments between times $t = 0$ and $t = 1$. Thus the total amount of the premium at time zero is needed to meet the benefit paying liabilities in this period and cover the initial expense. That is why we are talking about a full preliminary term method if the initial expense is defined as the Zillmer maximum.

It follows from (18) and (24) that under the full preliminary term method

$${}_1V^{*net} = 0. \tag{28}$$

Thus the modified annual net premiums payable at times $t = 1,2,\ldots,n - 1$, which are equal to the annual gross premium P'' (see (16)), will provide the money needed to pay the benefits after time $t = 1$. Thus

$$P'' = \frac{EPV_1\,(\text{cash flow of benefits after time } t = 1)}{\ddot{a}_{x+1:\overline{n-1}|}}. \tag{29}$$

We have seen in Section 5.1 that the whole life and term insurances and the endowment insurances satisfy the conditions of Theorem 1.1 of Section 5.1. Hence we can apply our results to these classes of insurances.

For example, in the case of an n year endowment insurance of \$1 with annual premiums, we have

$${}_1V^{net} = {}_1V_{x:\overline{n}|} = 1 - \frac{\ddot{a}_{x+1:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}}.$$

Thus

$$\frac{\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x+1:\overline{n-1}|}} = \frac{1}{1 - {}_1V_{x:\overline{n}|}}$$

so from (25), the Zillmer maximum is

$$I_{Zillmer} = \frac{{}_1V_{x:\overline{n}|}}{1 - {}_1V_{x:\overline{n}|}}. \tag{30}$$

Furthermore, using the full preliminary term method, (29) gives the following gross annual premium:

$$P'' = \frac{A_{x+1:\overline{n-1}|}}{\ddot{a}_{x+1:\overline{n-1}|}} = P_{x+1:\overline{n-1}|}. \tag{31}$$

Using (16) and (27) we get

$$P_0^* = A^{\,1}_{x:\overline{1}|} \tag{32}$$

and

$$P_t^* = P_{x+1:\overline{n-1}|}, \quad t = 1,2,\ldots,n - 1. \tag{33}$$

Furthermore, from (18) we obtain

$${}_tV^{mod} = {}_tV^{*net} = {}_{t-1}V_{x+1:\overline{n-1}|}, \quad t = 1,2,\ldots,n. \tag{34}$$

If we consider a whole life insurance of \$1 with annual premiums, we get

$${}_1V^{net} = {}_1V_x = 1 - \frac{\ddot{a}_{x+1}}{\ddot{a}_x}.$$

Then the Zillmer maximum is

$$I_{Zillmer} = \frac{{}_1V_x}{1 - {}_1V_x}. \tag{35}$$

Using the full preliminary term method, the annual gross premium is

$$P'' = \frac{A_{x+1}}{\ddot{a}_{x+1}} = P(A_{x+1}). \tag{36}$$

Note that using the notation P_{x+1} instead of $P(A_{x+1})$ would be confusing here. Moreover,

$$P_0^* = A^{\,1}_{x:\overline{1}|} \tag{37}$$

and

$$P_t^* = P(A_{x+1}), \quad t = 1,2,\ldots . \tag{38}$$

We also have

$$_tV^{mod} = {}_tV^{*net} = {}_{t-1}V(A_{x+1}), \quad t = 1,2,\ldots . \tag{39}$$

There are also other methods of defining the modified net premiums. They are usually of the form

$$P_0^* = \alpha$$

$$P_1^* = P_2^* = \ldots = P_{k-1}^* = \beta$$

and

$$P_k^* = P_{k+1}^* = \ldots = P_{n-1}^* = P.$$

Methods of this type are the commissioners reserve valuation method used in many U.S. states and the Canadian method.

Note that the full preliminary term method is a special kind of these methods with $k = n$,

$$P_0^* = EPV_0 \text{ (cash flow of benefits between times } t = 0 \text{ and } t = 1) = \alpha$$

and

$$P_1^* = \ldots = P_{n-1}^* = P'' = \beta.$$

EXAMPLE 3.1. The premiums for a 10 year pure endowment of \$5000 on a life aged 25 are payable annually. Determine the Zillmer maximum for the insurance. Assuming an initial expense of \$120 is paid from the first premium, find the modified net premiums. Also obtain the expressions for the modified net premium reserves at duration $t = 1,2,\ldots,10$ and evaluate them numerically at $t = 1$, $t = 8$, and $t = 10$. Base the computations on a 6% annual interest rate.

Solution: Denoting the net annual premium by P, we can write

$$P\,\ddot{a}_{25:\overline{10}|} = 5000\,A_{25:\overset{\;\;1}{\overline{10}|}}.$$

Now

$$\ddot{a}_{25:\overline{10}|} = \frac{N_{25} - N_{35}}{D_{25}} = \frac{361578.07 - 188663.76}{22286.35} = 7.75875$$

and

$$A^{\;\;1}_{25:\overline{10}|} = \frac{D_{35}}{D_{25}} = \frac{12256.76}{22286.35} = 0.5499671$$

so

$$P = 5000\frac{0.5499671}{7.75875} = \$354.42.$$

The net premium reserve at duration t is

$$_tV^{net} = \frac{1}{_tE_{25}}\, P\ddot{a}_{25:\overline{t}|} = \frac{D_{25}}{D_{25+t}}\, P\frac{N_{25} - N_{25+t}}{D_{25}} = P\frac{N_{25} - N_{25+t}}{D_{25+t}}, \quad t = 1,2,\ldots,10.$$

Using (25), the Zillmer maximum is

$$I_{Zillmer} = {_1V^{net}}\frac{\ddot{a}_{25:\overline{10}|}}{\ddot{a}_{26:\overline{9}|}}.$$

We have

$$_1V^{net} = P\frac{N_{25} - N_{26}}{D_{26}} = P\frac{D_{25}}{D_{26}} = 354.42\,\frac{22286.35}{20999.15} = 376.15$$

and

$$\ddot{a}_{26:\overline{9}|} = \frac{N_{26} - N_{35}}{D_{26}} = \frac{339291.72 - 188663.76}{20999.15} = 7.17305.$$

Thus

$$I_{Zillmer} = 376.15\;\frac{7.75875}{7.17305} = \$406.86.$$

If the initial expense is $\$120$, the gross annual premium is

$$P'' = P + \frac{I}{\ddot{a}_{25:\overline{10}|}} = 354.42 + \frac{120}{7.75875} = \$369.89.$$

Thus, the modified net premiums are

$$P_0^* = 369.89 - 120 = \$249.89 \text{ at time } t = 0$$

and

$$P_t^* = \$369.89 \text{ at time } t = 1,2,\ldots,9.$$

From (18) we get

$${}_tV^{*net} = {}_tV^{mod}, \quad t = 1,2,\ldots,10$$

and using (6), we get

$$\begin{aligned}
{}_tV^{*net} &= {}_tV^{net} - I\frac{\ddot{a}_{25+t:\overline{10-t}|}}{\ddot{a}_{25:\overline{10}|}} \\
&= P\frac{\ddot{a}_{25:\overline{5}|}}{{}_tE_{25}} - I\frac{\ddot{a}_{25+t:\overline{10-t}|}}{\ddot{a}_{25:\overline{10}|}} \\
&= 354.42\frac{\ddot{a}_{25:\overline{t}|}}{{}_tE_{25}} - 120\frac{\ddot{a}_{25+t:\overline{10-t}|}}{\ddot{a}_{25:\overline{10}|}}, \quad t = 1,2,\ldots,10.
\end{aligned}$$

Hence the modified net premium reserve at duration $t = 1$ is

$$\begin{aligned}
{}_1V^{*net} &= {}_1V^{net} - I\frac{\ddot{a}_{26:\overline{9}|}}{\ddot{a}_{25:\overline{10}|}} \\
&= 376.15 - 120\frac{7.17305}{7.75875} \\
&= 265.21.
\end{aligned}$$

For $t = 8$, we have

$${}_8V^{*net} = {}_8V^{net} - 120\ \frac{\ddot{a}_{33:\overline{2}|}}{\ddot{a}_{25:\overline{10}|}}.$$

Now

$$\begin{aligned}
{}_8V^{net} &= P\frac{N_{25} - N_{33}}{D_{33}} \\
&= 354.42\frac{361578.07 - 215503.30}{13822.67} \\
&= 3745.43,
\end{aligned}$$

and

$$\begin{aligned}
\ddot{a}_{33:\overline{2}|} &= \frac{N_{33} - N_{35}}{D_{33}} \\
&= \frac{215503.30 - 188663.76}{13822.67} \\
&= 1.94170.
\end{aligned}$$

Hence the modified net premium reserve at duration $t = 8$ is

$$_8V^{*net} = 3745.43 - 120\ \frac{1.94170}{7.75875}$$
$$= \$3715.40.$$

Using (8) and (18), the modified net premium reserve at duration $t = 10$ is

$$_{10}V^{*net} = {}_{10}V^{mod} = {}_{10}V^{net} = \$5000.$$

EXAMPLE 3.2. A whole life insurance of $8000 issued to a life aged 50 is purchased by annual premiums. Using the full preliminary term method, find the initial expenses and determine the modified net premiums. Also derive expressions for the modified net premium reserves at duration $t = 1,2,...$ and evaluate them numerically at $t = 1$, $t = 2$, and $t = 30$. Use a 6% annual rate of interest.

Solution: Let us denote the net annual premium by P. Then we have

$$P\,\ddot{a}_{50} = 8000\,A_{50}.$$

Now

$$\ddot{a}_{50} = 13.26683$$

and

$$A_{50} = 0.2490475$$

hence

$$P = 8000\ \frac{0.2490475}{13.26683} = \$150.18.$$

The initial expense is given by the Zillmer maximum. From (35), we obtain

$$I_{Zillmer} = 8000\ \frac{_1V_{50}}{1 - {}_1V_{50}}.$$

Since

$$_1V_{50} = 1 - \frac{\ddot{a}_{51}}{\ddot{a}_{50}} = 1 - \frac{13.08027}{13.26683} = 0.0140621,$$

the initial expense is

$$I_{Zillmer} = 8000\ \frac{0.0140621}{1 - 0.0140621} = \$114.10.$$

Therefore, the gross annual premium is

$$P'' = P + \frac{I}{\ddot{a}_{50}} = 150.18 + \frac{114.10}{13.26683} = \$158.78.$$

Another way of obtaining the gross annual premium is to use (36):

$$P'' = 8000\ P(A_{51}) = 8000\ \frac{A_{51}}{\ddot{a}_{51}} = 8000\ \frac{0.2596073}{13.08027} = \$158.78.$$

Therefore, the modified net premiums are

$$P_0^* = 158.78 - 144.10 = \$14.68 \text{ at time } t = 0$$

and

$$P_t^* = \$158.78 \text{ at time } t = 1,2,\dots .$$

Using (39), the modified net premium reserves are

$${}_tV^{*net} = 8000\ {}_{t\text{-}1}V(A_{51}) = 8000\left(1 - \frac{\ddot{a}_{50+t}}{\ddot{a}_{51}}\right), \quad t = 1,2,\dots .$$

The modified net premium reserve at duration $t = 1$ is

$${}_1V^{*net} = 8000\left(1 - \frac{\ddot{a}_{51}}{\ddot{a}_{51}}\right) = 0$$

giving the same result as (28).

Furthermore, we get

$${}_2V^{*net} = 8000\left(1 - \frac{\ddot{a}_{52}}{\ddot{a}_{51}}\right) = 8000\left(1 - \frac{12.88785}{13.08027}\right) = \$117.69$$

and

$${}_{30}V^{*net} = 8000\left(1 - \frac{\ddot{a}_{80}}{\ddot{a}_{51}}\right) = 8000\left(1 - \frac{5.90503}{13.08027}\right) = \$4388.44.$$

PROBLEMS

3.1. A 20 year endowment of $6000 issued to a life aged 40 is purchased by annual premiums. Obtain the Zillmer maximum for the

insurance. Find the modified net premiums if an initial expense of $100 is paid from the first premium. Furthermore, derive the expressions for the modified net premium reserves at duration $t = 1,2,\ldots,20$ and evaluate them numerically at $t = 1$, $t = 10$, and $t = 20$. Use a 6% annual rate of interest.

3.2. A 25 year term insurance of $8000 issued to a life aged 50 is purchased by annual premiums. Find the Zillmer maximum for the insurance. Compute the modified net premiums and find the expressions for the modified net premium reserves at duration $t = 1,2,\ldots,25$ if an initial expense of $50 is paid from the first premium. Determine the numerical value of the net premium reserves at duration $t = 1$, $t = 10$, and $t = 25$. Use a 6% annual rate of interest.

3.3. The premiums for a whole life insurance of $5000 issued to a life aged 40 are payable annually. Determine the initial expenses using the full preliminary term method. Obtain the modified net premiums. Find the expressions for the modified net premium reserves at duration $t = 1,2,\ldots$ and evaluate them numerically at $t = 1$, $t = 5$, and $t = 15$. Base the computations on a 6% annual interest rate.

Answers to Odd-Numbered Problems

Chapter 1

Section 1

1.1.	520, 500, 20			
1.3.	810.64, 10.64			
1.5.	3629.25			
1.7.	a) 0.00803	b) 0.02532	c) 0.03769	
1.9.	a) 0.00415, 0.04889			
	b) 0.02490, 0.04940			
	c) 0.03966, 0.04975, 0.04879			
1.11.	a) 2500.44	b) 2503.22	c) 2675	
1.13.	a) 33.83	b) 101.49	c) 8.46	
1.15.	a) 32	b) 30.77		
1.17.	a) 0.00786, 0.04860			
	b) 0.01222, 0.04849			
	c) 0.01762, 0.04836			

Section 2

2.1.	2938.50			
2.3.	0.04373, yes			
2.5.	152.78, 162.08			
2.7.	b) 116.60, 483.40			
	c) 64.56, 605.44			

Section 3

3.1.	a) 1	b) 7.8017	c) 13.9716	d) 38.9927
3.3.	a) 6.2915	b) 4.5782		
3.5.	1575.49			
3.7.	a) 6409.19	b) 7797.76		

3.9. a) 0.9615 b) 10.5631 c) 18.2919 d) 41.6459

3.11. 1281.39

3.13. 7685.37

3.15. 5646.29

3.17. a) 9.8558 b) 37.7866 c) 9.3105 d) 26.6312
e) 192.9002

3.19. 1005.37

3.21. 50347.84, 120661.53

3.23. 16544.11, 24489.25

3.25. a) 12161.70 b) 11582.55 c) 11893.90 d) 11845.65
e) 11869.76

3.27. 1720.17

Chapter 2

Section 1

1.1 0.53526, 0.0125

1.3. Hint: Prove that $\frac{d}{dt}(\log S(t)) = c$, where c is a constant and integrate both sides of the equation.

Section 2

2.1. a) 0.9970045 b) 0.0029955 c) 0.9792190
d) 0.0295545 e) 0.0147105

2.3. Hint: Show that $\frac{d}{dx}\log \ell_x = -\frac{1}{\omega - x}$ and integrate both sides of the equation.

Section 3

3.1. a) 0.9940801 b) 0.0015289 c) 0.9647228
d) 0.2166647 e) 0.1236632

3.3. a) 94912.10 b) 0.9991946 c) 0.9653463

d) 0.0793169

3.5. a) 0.00132 b) 0.99823 c) 0.99809

d) 0.99083 e) 0.01170 f) 0.00785

Chapter 3

Section 1

1.1
$$C(t^*,t) = \begin{cases} 500 & \text{if } t^* = 5 \text{ and } t^* < t \\ 700 & \text{if } t^* = 10 \text{ and } t^* < t \\ 1000 & \text{if } t^* = 15 \text{ and } t^* < t \\ 0 & \text{otherwise} \end{cases}$$

$$g(t) = 500 \frac{1}{1.04^5} I(5 < t) + 700 \frac{1}{1.04^{10}} I(10 < t) + 1000 \frac{1}{1.04^{15}} I(15 < \mathrm{t})$$

a) 0 b) 883.86 c) 1439.12

1.3. a)
$$C(t^*,t) = \begin{cases} 5000 & \text{if } t^* = t \\ 0 & \text{otherwise} \end{cases}$$

$$g(t) = 5000 \frac{1}{1.04^t}$$

$g(17.2) = 2546.81$

b)
$$C(t^*,t) = \begin{cases} 5000 & \text{if } t^* = [t] + 1 \\ 0 & \text{otherwise} \end{cases}$$

$$g(t) = 5000 \frac{1}{1.04^{[t]+1}}$$

$g(17.2) = 2468.14$

1.5. 1406.55

1.7. 4098.83

Section 2

2.1. a) 0.8087671 b) 23873.40 c) 0.6653682

2.3. 1180.07, 237.40

Section 3

3.1. a) 51.8757 b) 71.6937 c) 0.0029553

3.3. 2583.92, 1420.04

3.5.	3515.64		
3.7.	133.05		
3.9.	3371.60		
3.11.	1586.41, 1068.56		
3.13.	a) 13857.19	b) 4.2967280	c) 0.1783632

Section 4

4.1.	a) 0.5615188	b) 0.0007724	c) 0.1489294
4.3.	a) 0.2580743	b) 0.0086869	c) 0.1147878

Section 5

5.1.	Hint: Use algebra or general reasoning.		
5.3.	31712.24, 4299.71		
5.5.	4583.03		
5.7.	a) 1702518.15	b) 206.43624	c) 68.89061
5.9.	Hint: Use algebra or general reasoning.		
5.11.	a) 14.39262	b) 5.7165073	c) 7.30872
	d) 0.2416486	e) 2.78121	f) 5.29950
5.13.	13963.89		
5.15.	40182.43		
5.17.	a) 15.39779	b) 11.48906	c) 8.32795
	d) 5.28064		
5.19.	69332.82		
5.21.	183570.14		
5.23.	a) 12.51683	b) 12.76236	c) 4.55475
	d) 1.26013		
5.25.	82202.47		
5.27.	43933.33		
5.31.	51539.80, 9564.16		
5.33.	10875.42		
5.35.	116630.52		

5.37. 12023.70

5.39. 824.51

5.41. 131603.78

Chapter 4

Section 1

1.1. 679.57

1.3. 2228.50

1.5. a) 33.63 b) 34.74

1.7. 66.42

1.9. 8.39

1.11. 201.16

1.13. 1875.36

1.15. a) 274.23 b) 251.62 c) 236.57
d) 235.18

Section 2

2.1. a) 300.66 b) 461.27 c) 160.61

2.3. 10070.25, 324.78

2.5. a) 123.66 b) 165.37

Chapter 5

Section 1

1.1 $P = 587.48$

$$_tV^{prosp} = 2000\,\frac{D_{50}}{D_{30+t}}$$

$$_tV^{retro} = \frac{P}{_tE_{30}} = \frac{D_{30}}{D_{30+t}}\,P$$

$_6V = 842.19$

$_{20}V = 1761.34$

1.3. $P = 1691.45$

$${}_tV^{prosp} = 4000\,\frac{(1.06)^{\frac{1}{2}}\,(M_{25+t} - M_{40}) + D_{40}}{D_{25+t}}$$

$${}_tV^{retro} = \frac{D_{25}}{D_{25+t}}\left(P - (1.06)^{\frac{1}{2}}\,\frac{M_{25} - M_{25+t}}{D_{25}}\right)$$

${}_5V = 2247.75$

${}_8V = 2668.94$

1.5. $P = 1653.53$

$${}_tV^{prosp} = 6000\left(a_{\overline{10-t}|} - \frac{N_{51+t} - N_{61}}{D_{50+t}}\right)$$

$${}_tV^{retro} = \frac{D_{50}}{D_{50+t}}$$

$$\left(P - 6000\left(a_{\overline{t-1}|} - \frac{N_{51} - N_{50+t}}{D_{50}}\right) + \left(v^t - \frac{D_{50+t}}{D_{50}}\right)(1 + a_{\overline{10-t}|})\right)$$

${}_5V = 723.16$

1.7. Hint: Use the formulas

$${}_tV_x = 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}$$

and

$${}_tV_{x:\overline{n}|} = 1 - \frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}}.$$

1.9. a) $P = 49.12$

$${}_tV^{prosp} = 2000\,\frac{D_{50}}{D_{30+t}} - P\,\frac{N_{30+t} - N_{50}}{D_{30+t}}$$

$${}_tV^{retro} = P\,\frac{N_{30} - N_{30+t}}{D_{30+t}}$$

${}_6V = 365.64$

${}_{18}V = 1665.98$

b) $P = 75.84$

$${}_tV^{prosp} = \begin{cases} 2000\,\dfrac{D_{50}}{D_{30+t}} - P\,\dfrac{N_{30+t} - N_{40}}{D_{30+t}} & \text{if } t \le 10 \\[2ex] 2000\,\dfrac{D_{50}}{D_{30+t}} & \text{if } t > 10 \end{cases}$$

$$ {}_tV^{retro} = \begin{cases} P\dfrac{N_{30} - N_{30+t}}{D_{30+t}} & \text{if } t \le 10 \\ P\dfrac{N_{30} - N_{40}}{D_{30+t}} & \text{if } t > 10 \end{cases} $$

${}_6V = 564.53$

${}_{18}V = 1761.39$

1.11. $P = 170.65$

${}_tV^{prosp}$

$$ = 4000\frac{(1.06)^{\frac{1}{2}}(M_{25+t} - M_{40}) + D_{40}}{D_{25+t}} - P\left(\frac{N_{25+t} - N_{40}}{D_{25+t}} - \frac{1}{2}\left(1 - \frac{D_{40}}{D_{25+t}}\right)\right) $$

${}_tV^{retro}$

$$ = \frac{D_{25}}{D_{25+t}}\left(P\left(\frac{N_{25} - N_{25+t}}{D_{25}} - \frac{1}{2}\left(1 - \frac{D_{25+t}}{D_{25}}\right)\right) - 4000 \cdot (1.06)^{\frac{1}{2}}\frac{M_{25} - M_{25+t}}{D_{25}}\right) $$

${}_5V = 964.44$

${}_8V = 1694.09$

1.13. a) 1926.82 b) 3372.00

1.15. 7

1.17. a) 440.59, 520.28 b) 680.25, 803.29

Section 2

2.1. mortality loss of 10869.21

2.3. mortality profit of 2428.09

2.5. mortality profit of 776.94

Section 3

3.1. $I_{Zillmer} = 169.17$

$P_0^* = 79.03$

$P_t^* = 179.03, \quad t = 1,2,\ldots,19$

$${}_tV^{*net} = 6000 - 6100\,\frac{\ddot{a}_{40+t:\overline{20-t}|}}{\ddot{a}_{40:\overline{20}|}}, \quad t = 1,2,\ldots,20$$

${}_1V^{*net} = 67.28$

$_{10}V^{*net} = 2071.89$

$_{20}V^{*net} = 6000$

3.3. $I = I_{Zillmer} = 44.31$

$P_0^* = 13.12$

$P_t^* = 57.43, \quad t = 1,2,3,...$

$$_tV^{*net} = 5000\left(1 - \frac{\ddot{a}_{40+t}}{\ddot{a}_{41}}\right), \quad t = 1,2,...$$

$_1V^{*net} = 0$

$_5V^{*net} = 195.54$

$_{15}V^{*net} = 820.70$

APPENDIX 1

COMPOUND INTEREST TABLES

COMPOUND INTEREST TABLES

1 per cent

	n	$(1+i)^n$	v^n	$a_{\overline{n}\rceil}$	$s_{\overline{n}\rceil}$
i = 0.010000	1	1.01000	0.99010	0.9901	1.0000
$i^{(2)}$ = 0.009975	2	1.02010	0.98030	1.9704	2.0100
$i^{(4)}$ = 0.009963	3	1.03030	0.97059	2.9410	3.0301
$i^{(12)}$ = 0.009954	4	1.04060	0.96098	3.9020	4.0604
δ = 0.009950	5	1.05101	0.95147	4.8534	5.1010
$(1+i)^{1/2}$ = 1.004988	6	1.06152	0.94205	5.7955	6.1520
$(1+i)^{1/4}$ = 1.002491	7	1.07214	0.93272	6.7282	7.2135
$(1+i)^{1/12}$ = 1.000830	8	1.08286	0.92348	7.6517	8.2857
v = 0.990099	9	1.09369	0.91434	8.5660	9.3685
$v^{1/2}$ = 0.995037	10	1.10462	0.90529	9.4713	10.4622
$v^{1/4}$ = 0.997516	11	1.11567	0.89632	10.3676	11.5668
$v^{1/12}$ = 0.999171	12	1.12683	0.88745	11.2551	12.6825
d = 0.009901	13	1.13809	0.87866	12.1337	13.8093
$d^{(2)}$ = 0.009926	14	1.14947	0.86996	13.0037	14.9474
$d^{(4)}$ = 0.009938	15	1.16097	0.86135	13.8651	16.0969
$d^{(12)}$ = 0.009946	16	1.17258	0.85282	14.7179	17.2579
$i/i^{(2)}$ = 1.002494	17	1.18430	0.84438	15.5623	18.4304
$i/i^{(4)}$ = 1.003742	18	1.19615	0.83602	16.3983	19.6147
$i/i^{(12)}$ = 1.004575	19	1.20811	0.82774	17.2260	20.8109
i/δ = 1.004992	20	1.22019	0.81954	18.0456	22.0190
$i/d^{(2)}$ = 1.007494	21	1.23239	0.81143	18.8570	23.2392
$i/d^{(4)}$ = 1.006242	22	1.24472	0.80340	19.6604	24.4716
$i/d^{(12)}$ = 1.005408	23	1.25716	0.79544	20.4558	25.7163
	24	1.26973	0.78757	21.2434	26.9735
	25	1.28243	0.77977	22.0232	28.2432
	26	1.29526	0.77205	22.7952	29.5256
	27	1.30821	0.76440	23.5596	30.8209
	28	1.32129	0.75684	24.3164	32.1291
	29	1.33450	0.74934	25.0658	33.4504
	30	1.34785	0.74192	25.8077	34.7849
	31	1.36133	0.73458	26.5423	36.1327
	32	1.37494	0.72730	27.2696	37.4941
	33	1.38869	0.72010	27.9897	38.8690
	34	1.40258	0.71297	28.7027	40.2577
	35	1.41660	0.70591	29.4086	41.6603
	36	1.43077	0.69892	30.1075	43.0769
	37	1.44508	0.69200	30.7995	44.5076
	38	1.45953	0.68515	31.4847	45.9527
	39	1.47412	0.67837	32.1630	47.4123
	40	1.48886	0.67165	32.8347	48.8864
	41	1.50375	0.66500	33.4997	50.3752
	42	1.51879	0.65842	34.1581	51.8790
	43	1.53398	0.65190	34.8100	53.3978
	44	1.54932	0.64545	35.4555	54.9318
	45	1.56481	0.63905	36.0945	56.4811
	46	1.58046	0.63273	36.7272	58.0459
	47	1.59626	0.62646	37.3537	59.6263
	48	1.61223	0.62026	37.9740	61.2226
	49	1.62835	0.61412	38.5881	62.8348
	50	1.64463	0.60804	39.1961	64.4632
	60	1.81670	0.55045	44.9550	81.6697
	70	2.00676	0.49831	50.1685	100.6763
	80	2.21672	0.45112	54.8882	121.6715
	90	2.44863	0.40839	59.1609	144.8633
	100	2.70481	0.36971	63.0289	170.4814

Source: Adapted from *Formulae and Tables for Actuarial Examinations*, published by the Faculty and Institute of Actuaries, 1980.

1.5 per cent

$i = 0.015000$
$i^{(2)} = 0.014944$
$i^{(4)} = 0.014916$
$i^{(12)} = 0.014898$
$\delta = 0.014889$
$(1+i)^{1/2} = 1.007472$
$(1+i)^{1/4} = 1.003729$
$(1+i)^{1/12} = 1.001241$
$v = 0.985222$
$v^{1/2} = 0.992583$
$v^{1/4} = 0.996285$
$v^{1/12} = 0.998760$
$d = 0.014778$
$d^{(2)} = 0.014833$
$d^{(4)} = 0.014861$
$d^{(12)} = 0.014879$
$i/i^{(2)} = 1.003736$
$i/i^{(4)} = 1.005608$
$i/i^{(12)} = 1.006857$
$i/\delta = 1.007481$
$i/d^{(2)} = 1.011236$
$i/d^{(4)} = 1.009358$
$i/d^{(12)} = 1.008107$

n	$(1+i)^n$	v^n	$a_{\overline{n}\rceil}$	$s_{\overline{n}\rceil}$
1	1.01500	0.98522	0.9852	1.0000
2	1.03023	0.97066	1.9559	2.0150
3	1.04568	0.95632	2.9122	3.0452
4	1.06136	0.94218	3.8544	4.0909
5	1.07728	0.92826	4.7826	5.1523
6	1.09344	0.91454	5.6972	6.2296
7	1.10984	0.90103	6.5982	7.3230
8	1.12649	0.88771	7.4859	8.4328
9	1.14339	0.87459	8.3605	9.5593
10	1.16054	0.86167	9.2222	10.7027
11	1.17795	0.84893	10.0711	11.8633
12	1.19562	0.83639	10.9075	13.0412
13	1.21355	0.82403	11.7315	14.2368
14	1.23176	0.81185	12.5434	15.4504
15	1.25023	0.79985	13.3432	16.6821
16	1.26899	0.78803	14.1313	17.9324
17	1.28802	0.77639	14.9076	19.2014
18	1.30734	0.76491	15.6726	20.4894
19	1.32695	0.75361	16.4262	21.7967
20	1.34686	0.74247	17.1686	23.1237
21	1.36706	0.73150	17.9001	24.4705
22	1.38756	0.72069	18.6208	25.8376
23	1.40838	0.71004	19.3309	27.2251
24	1.42950	0.69954	20.0304	28.6335
25	1.45095	0.68921	20.7196	30.0630
26	1.47271	0.67902	21.3986	31.5140
27	1.49480	0.66899	22.0676	32.9867
28	1.51722	0.65910	22.7267	34.4815
29	1.53998	0.64936	23.3761	35.9987
30	1.56308	0.63976	24.0158	37.5387
31	1.58653	0.63031	24.6461	39.1018
32	1.61032	0.62099	25.2671	40.6883
33	1.63448	0.61182	25.8790	42.2986
34	1.65900	0.60277	26.4817	43.9331
35	1.68388	0.59387	27.0756	45.5921
36	1.70914	0.58509	27.6607	47.2760
37	1.73478	0.57644	28.2371	48.9851
38	1.76080	0.56792	28.8051	50.7199
39	1.78721	0.55953	29.3646	52.4807
40	1.81402	0.55126	29.9158	54.2679
41	1.84123	0.54312	30.4590	56.0819
42	1.86885	0.53509	30.9941	57.9231
43	1.89688	0.52718	31.5212	59.7920
44	1.92533	0.51939	32.0406	61.6889
45	1.95421	0.51171	32.5523	63.6142
46	1.98353	0.50415	33.0565	65.5684
47	2.01328	0.49670	33.5532	67.5519
48	2.04348	0.48936	34.0426	69.5652
49	2.07413	0.48213	34.5247	71.6087
50	2.10524	0.47500	34.9997	73.6828
60	2.44322	0.40930	39.3803	96.2147
70	2.83546	0.35268	43.1549	122.3638
80	3.29066	0.30389	46.4073	152.7109
90	3.81895	0.26185	49.2099	187.9299
100	4.43205	0.22563	51.6247	228.8030

2 per cent

i	0.020000
$i^{(2)}$	0.019901
$i^{(4)}$	0.019852
$i^{(12)}$	0.019819
δ	0.019803
$(1+i)^{1/2}$	1.009950
$(1+i)^{1/4}$	1.004963
$(1+i)^{1/12}$	1.001652
v	0.980392
$v^{1/2}$	0.990148
$v^{1/4}$	0.995062
$v^{1/12}$	0.998351
d	0.019608
$d^{(2)}$	0.019705
$d^{(4)}$	0.019754
$d^{(12)}$	0.019786
$i/i^{(2)}$	1.004975
$i/i^{(4)}$	1.007469
$i/i^{(12)}$	1.009134
i/δ	1.009967
$i/d^{(2)}$	1.014975
$i/d^{(4)}$	1.012469
$i/d^{(12)}$	1.010801

n	$(1+i)^n$	v^n	$a_{\overline{n}\rceil}$	$s_{\overline{n}\rceil}$
1	1.02000	0.98039	0.9804	1.0000
2	1.04040	0.96117	1.9416	2.0200
3	1.06121	0.94232	2.8839	3.0604
4	1.08243	0.92385	3.8077	4.1216
5	1.10408	0.90573	4.7135	5.2040
6	1.12616	0.88797	5.6014	6.3081
7	1.14869	0.87056	6.4720	7.4343
8	1.17166	0.85349	7.3255	8.5830
9	1.19509	0.83676	8.1622	9.7546
10	1.21899	0.82035	8.9826	10.9497
11	1.24337	0.80426	9.7868	12.1687
12	1.26824	0.78849	10.5753	13.4121
13	1.29361	0.77303	11.3484	14.6803
14	1.31948	0.75788	12.1062	15.9739
15	1.34587	0.74301	12.8493	17.2934
16	1.37279	0.72845	13.5777	18.6393
17	1.40024	0.71416	14.2919	20.0121
18	1.42825	0.70016	14.9920	21.4123
19	1.45681	0.68643	15.6785	22.8406
20	1.48595	0.67297	16.3514	24.2974
21	1.51567	0.65978	17.0112	25.7833
22	1.54598	0.64684	17.6580	27.2990
23	1.57690	0.63416	18.2922	28.8450
24	1.60844	0.62172	18.9139	30.4219
25	1.64061	0.60953	19.5235	32.0303
26	1.67342	0.59758	20.1210	33.6709
27	1.70689	0.58586	20.7069	35.3443
28	1.74102	0.57437	21.2813	37.0512
29	1.77584	0.56311	21.8444	38.7922
30	1.81136	0.55207	22.3965	40.5681
31	1.84759	0.54125	22.9377	42.3794
32	1.88454	0.53063	23.4683	44.2270
33	1.92223	0.52023	23.9886	46.1116
34	1.96068	0.51003	24.4986	48.0338
35	1.99989	0.50003	24.9986	49.9945
36	2.03989	0.49022	25.4888	51.9944
37	2.08069	0.48061	25.9695	54.0343
38	2.12230	0.47119	26.4406	56.1149
39	2.16474	0.46195	26.9026	58.2372
40	2.20804	0.45289	27.3555	60.4020
41	2.25220	0.44401	27.7995	62.6100
42	2.29724	0.43530	28.2348	64.8622
43	2.34319	0.42677	28.6616	67.1595
44	2.39005	0.41840	29.0800	69.5027
45	2.43785	0.41020	29.4902	71.8927
46	2.48661	0.40215	29.8923	74.3306
47	2.53634	0.39427	30.2866	76.8172
48	2.58707	0.38654	30.6731	79.3535
49	2.63881	0.37896	31.0521	81.9406
50	2.69159	0.37153	31.4236	84.5794
60	3.28103	0.30478	34.7609	114.0515
70	3.99956	0.25003	37.4986	149.9779
80	4.87544	0.20511	39.7445	193.7720
90	5.94313	0.16826	41.5869	247.1567
100	7.24465	0.13803	43.0984	312.2323

2.5 per cent

$i = 0.025000$
$i^{(2)} = 0.024846$
$i^{(4)} = 0.024769$
$i^{(12)} = 0.024718$
$\delta = 0.024693$
$(1+i)^{1/2} = 1.012423$
$(1+i)^{1/4} = 1.006192$
$(1+i)^{1/12} = 1.002060$
$v = 0.975610$
$v^{1/2} = 0.987730$
$v^{1/4} = 0.993846$
$v^{1/12} = 0.997944$
$d = 0.024390$
$d^{(2)} = 0.024541$
$d^{(4)} = 0.024617$
$d^{(12)} = 0.024667$
$i/i^{(2)} = 1.006211$
$i/i^{(4)} = 1.009327$
$i/i^{(12)} = 1.011407$
$i/\delta = 1.012449$
$i/d^{(2)} = 1.018711$
$i/d^{(4)} = 1.015577$
$i/d^{(12)} = 1.013491$

n	$(1+i)^n$	v^n	$a_{\overline{n}\rceil}$	$s_{\overline{n}\rceil}$
1	1.02500	0.97561	0.9756	1.0000
2	1.05063	0.95181	1.9274	2.0250
3	1.07689	0.92860	2.8560	3.0756
4	1.10381	0.90595	3.7620	4.1525
5	1.13141	0.88385	4.6458	5.2563
6	1.15969	0.86230	5.5081	6.3877
7	1.18869	0.84127	6.3494	7.5474
8	1.21840	0.82075	7.1701	8.7361
9	1.24886	0.80073	7.9709	9.9545
10	1.28008	0.78120	8.7521	11.2034
11	1.31209	0.76214	9.5142	12.4835
12	1.34489	0.74356	10.2578	13.7956
13	1.37851	0.72542	10.9832	15.1404
14	1.41297	0.70773	11.6909	16.5190
15	1.44830	0.69047	12.3814	17.9319
16	1.48451	0.67362	13.0550	19.3802
17	1.52162	0.65720	13.7122	20.8647
18	1.55966	0.64117	14.3534	22.3863
19	1.59865	0.62553	14.9789	23.9460
20	1.63862	0.61027	15.5892	25.5447
21	1.67958	0.59539	16.1845	27.1833
22	1.72157	0.58086	16.7654	28.8629
23	1.76461	0.56670	17.3321	30.5844
24	1.80873	0.55288	17.8850	32.3490
25	1.85394	0.53939	18.4244	34.1578
26	1.90029	0.52623	18.9506	36.0117
27	1.94780	0.51340	19.4640	37.9120
28	1.99650	0.50088	19.9649	39.8598
29	2.04641	0.48866	20.4535	41.8563
30	2.09757	0.47674	20.9303	43.9027
31	2.15001	0.46511	21.3954	46.0003
32	2.20376	0.45377	21.8492	48.1503
33	2.25885	0.44270	22.2919	50.3540
34	2.31532	0.43191	22.7238	52.6129
35	2.37321	0.42137	23.1452	54.9282
36	2.43254	0.41109	23.5563	57.3014
37	2.49335	0.40107	23.9573	59.7339
38	2.55568	0.39128	24.3486	62.2273
39	2.61957	0.38174	24.7303	64.7830
40	2.68506	0.37243	25.1028	67.4026
41	2.75219	0.36335	25.4661	70.0876
42	2.82100	0.35448	25.8206	72.8398
43	2.89152	0.34584	26.1664	75.6608
44	2.96381	0.33740	26.5038	78.5523
45	3.03790	0.32917	26.8330	81.5161
46	3.11385	0.32115	27.1542	84.5540
47	3.19170	0.31331	27.4675	87.6679
48	3.27149	0.30567	27.7732	90.8596
49	3.35328	0.29822	28.0714	94.1311
50	3.43711	0.29094	28.3623	97.4843
60	4.39979	0.22728	30.9087	135.9916
70	5.63210	0.17755	32.8979	185.2841
80	7.20957	0.13870	34.4518	248.3827
90	9.22886	0.10836	35.6658	329.1543
100	11.81372	0.08465	36.6141	432.5487

3 per cent

$i = 0.030000$
$i^{(2)} = 0.029778$
$i^{(4)} = 0.029668$
$i^{(12)} = 0.029595$
$\delta = 0.029559$
$(1+i)^{1/2} = 1.014889$
$(1+i)^{1/4} = 1.007417$
$(1+i)^{1/12} = 1.002466$
$v = 0.970874$
$v^{1/2} = 0.985329$
$v^{1/4} = 0.992638$
$v^{1/12} = 0.997540$
$d = 0.029126$
$d^{(2)} = 0.029341$
$d^{(4)} = 0.029450$
$d^{(12)} = 0.029522$
$i/i^{(2)} = 1.007445$
$i/i^{(4)} = 1.011181$
$i/i^{(12)} = 1.013677$
$i/\delta = 1.014926$
$i/d^{(2)} = 1.022445$
$i/d^{(4)} = 1.018681$
$i/d^{(12)} = 1.016177$

n	$(1+i)^n$	v^n	$a_{\overline{n}\rceil}$	$s_{\overline{n}\rceil}$
1	1.03000	0.97087	0.9709	1.0000
2	1.06090	0.94260	1.9135	2.0300
3	1.09273	0.91514	2.8286	3.0909
4	1.12551	0.88849	3.7171	4.1836
5	1.15927	0.86261	4.5797	5.3091
6	1.19405	0.83748	5.4172	6.4684
7	1.22987	0.81309	6.2303	7.6625
8	1.26677	0.78941	7.0197	8.8923
9	1.30477	0.76642	7.7861	10.1591
10	1.34392	0.74409	8.5302	11.4639
11	1.38423	0.72242	9.2526	12.8078
12	1.42576	0.70138	9.9540	14.1920
13	1.46853	0.68095	10.6350	15.6178
14	1.51259	0.66112	11.2961	17.0863
15	1.55797	0.64186	11.9379	18.5989
16	1.60471	0.62317	12.5611	20.1569
17	1.65285	0.60502	13.1661	21.7616
18	1.70243	0.58739	13.7535	23.4144
19	1.75351	0.57029	14.3238	25.1169
20	1.80611	0.55368	14.8775	26.8704
21	1.86029	0.53755	15.4150	28.6765
22	1.91610	0.52189	15.9369	30.5368
23	1.97359	0.50669	16.4436	32.4529
24	2.03279	0.49193	16.9355	34.4265
25	2.09378	0.47761	17.4131	36.4593
26	2.15659	0.46369	17.8768	38.5530
27	2.22129	0.45019	18.3270	40.7096
28	2.28793	0.43708	18.7641	42.9309
29	2.35657	0.42435	19.1885	45.2189
30	2.42726	0.41199	19.6004	47.5754
31	2.50008	0.39999	20.0004	50.0027
32	2.57508	0.38834	20.3888	52.5028
33	2.65234	0.37703	20.7658	55.0778
34	2.73191	0.36604	21.1318	57.7302
35	2.81386	0.35538	21.4872	60.4621
36	2.89828	0.34503	21.8323	63.2759
37	2.98523	0.33498	22.1672	66.1742
38	3.07478	0.32523	22.4925	69.1594
39	3.16703	0.31575	22.8082	72.2342
40	3.26204	0.30656	23.1148	75.4013
41	3.35990	0.29763	23.4124	78.6633
42	3.46070	0.28896	23.7014	82.0232
43	3.56452	0.28054	23.9819	85.4839
44	3.67145	0.27237	24.2543	89.0484
45	3.78160	0.26444	24.5187	92.7199
46	3.89504	0.25674	24.7754	96.5015
47	4.01190	0.24926	25.0247	100.3965
48	4.13225	0.24200	25.2667	104.4084
49	4.25622	0.23495	25.5017	108.5406
50	4.38391	0.22811	25.7298	112.7969
60	5.89160	0.16973	27.6756	163.0534
70	7.91782	0.12630	29.1234	230.5941
80	10.64089	0.09398	30.2008	321.3630
90	14.30047	0.06993	31.0024	443.3489
100	19.21863	0.05203	31.5989	607.2877

3.5 per cent

$i = 0.03500$
$i^{(2)} = 0.034699$
$i^{(4)} = 0.034550$
$i^{(12)} = 0.034451$
$\delta = 0.034401$
$(1+i)^{1/2} = 1.017349$
$(1+i)^{1/4} = 1.008637$
$(1+i)^{1/12} = 1.002871$
$v = 0.966184$
$v^{1/2} = 0.982946$
$v^{1/4} = 0.991437$
$v^{1/12} = 0.997137$
$d = 0.033816$
$d^{(2)} = 0.034107$
$d^{(4)} = 0.034254$
$d^{(12)} = 0.034352$
$i/i^{(2)} = 1.008675$
$i/i^{(4)} = 1.013031$
$i/i^{(12)} = 1.015942$
$i/\delta = 1.017400$
$i/d^{(2)} = 1.026175$
$i/d^{(4)} = 1.021781$
$i/d^{(12)} = 1.018859$

n	$(1+i)^n$	v^n	$a_{\overline{n}\rvert}$	$s_{\overline{n}\rvert}$
1	1.03500	0.96618	0.9662	1.0000
2	1.07123	0.93351	1.8997	2.0350
3	1.10872	0.90194	2.8016	3.1062
4	1.14752	0.87144	3.6731	4.2149
5	1.18769	0.84197	4.5151	5.3625
6	1.22926	0.81350	5.3286	6.5502
7	1.27228	0.78599	6.1145	7.7794
8	1.31681	0.75941	6.8740	9.0517
9	1.36290	0.73373	7.6077	10.3685
10	1.41060	0.70892	8.3166	11.7314
11	1.45997	0.68495	9.0016	13.1420
12	1.51107	0.66178	9.6633	14.6020
13	1.56396	0.63940	10.3027	16.1130
14	1.61869	0.61778	10.9205	17.6770
15	1.67535	0.59689	11.5174	19.2957
16	1.73399	0.57671	12.0941	20.9710
17	1.79468	0.55720	12.6513	22.7050
18	1.85749	0.53836	13.1897	24.4997
19	1.92250	0.52016	13.7098	26.3572
20	1.98979	0.50257	14.2124	28.2797
21	2.05943	0.48557	14.6980	30.2695
22	2.13151	0.46915	15.1671	32.3289
23	2.20611	0.45329	15.6204	34.4604
24	2.28333	0.43796	16.0584	36.6665
25	2.36324	0.42315	16.4815	38.9499
26	2.44596	0.40884	16.8904	41.3131
27	2.53157	0.39501	17.2854	43.7591
28	2.62017	0.38165	17.6670	46.2906
29	2.71188	0.36875	18.0358	48.9108
30	2.80679	0.35628	18.3920	51.6227
31	2.90503	0.34423	18.7363	54.4295
32	3.00671	0.33259	19.0689	57.3345
33	3.11194	0.32134	19.3902	60.3412
34	3.22086	0.31048	19.7007	63.4532
35	3.33359	0.29998	20.0007	66.6740
36	3.45027	0.28983	20.2905	70.0076
37	3.57103	0.28003	20.5705	73.4579
38	3.69601	0.27056	20.8411	77.0289
39	3.82537	0.26141	21.1025	80.7249
40	3.95926	0.25257	21.3551	84.5503
41	4.09783	0.24403	21.5991	88.5095
42	4.24126	0.23578	21.8349	92.6074
43	4.38970	0.22781	22.0627	96.8486
44	4.54334	0.22010	22.2828	101.2383
45	4.70236	0.21266	22.4955	105.7817
46	4.86694	0.20547	22.7009	110.4840
47	5.03728	0.19852	22.8994	115.3510
48	5.21359	0.19181	23.0912	120.3883
49	5.39606	0.18532	23.2766	125.6018
50	5.58493	0.17905	23.4556	130.9979
60	7.87809	0.12693	24.9447	196.5169
70	11.11283	0.08999	26.0004	288.9379
80	15.67574	0.06379	26.7488	419.3068
90	22.11218	0.04522	27.2793	603.2050
100	31.19141	0.03206	27.6554	862.6117

4 per cent

$i = 0.04000$
$i^{(2)} = 0.039608$
$i^{(4)} = 0.039414$
$i^{(12)} = 0.039285$
$\delta = 0.039221$
$(1+i)^{1/2} = 1.019804$
$(1+i)^{1/4} = 1.009853$
$(1+i)^{1/12} = 1.003274$
$v = 0.961538$
$v^{1/2} = 0.980581$
$v^{1/4} = 0.990243$
$v^{1/12} = 0.996737$
$d = 0.038462$
$d^{(2)} = 0.038839$
$d^{(4)} = 0.039029$
$d^{(12)} = 0.039157$
$i/i^{(2)} = 1.009902$
$i/i^{(4)} = 1.014877$
$i/i^{(12)} = 1.018204$
$i/\delta = 1.019869$
$i/d^{(2)} = 1.029902$
$i/d^{(4)} = 1.024877$
$i/d^{(12)} = 1.021537$

n	$(1+i)^n$	v^n	$a_{\overline{n}\rceil}$	$s_{\overline{n}\rceil}$
1	1.04000	0.96154	0.9615	1.0000
2	1.08160	0.92456	1.8861	2.0400
3	1.12486	0.88900	2.7751	3.1216
4	1.16986	0.85480	3.6299	4.2465
5	1.21665	0.82193	4.4518	5.4163
6	1.26532	0.79031	5.2421	6.6330
7	1.31593	0.75992	6.0021	7.8983
8	1.36857	0.73069	6.7327	9.2142
9	1.42331	0.70259	7.4353	10.5828
10	1.48024	0.67556	8.1109	12.0061
11	1.53945	0.64958	8.7605	13.4864
12	1.60103	0.62460	9.3851	15.0258
13	1.66507	0.60057	9.9856	16.6268
14	1.73168	0.57748	10.5631	18.2919
15	1.80094	0.55526	11.1184	20.0236
16	1.87298	0.53391	11.6523	21.8245
17	1.94790	0.51337	12.1657	23.6975
18	2.02582	0.49363	12.6593	25.6454
19	2.10685	0.47464	13.1339	27.6712
20	2.19112	0.45639	13.5903	29.7781
21	2.27877	0.43883	14.0292	31.9692
22	2.36992	0.42196	14.4511	34.2480
23	2.46472	0.40573	14.8568	36.6179
24	2.56330	0.39012	15.2470	39.0826
25	2.66584	0.37512	15.6221	41.6459
26	2.77247	0.36069	15.9828	44.3117
27	2.88337	0.34682	16.3296	47.0842
28	2.99870	0.33348	16.6631	49.9676
29	3.11865	0.32065	16.9837	52.9663
30	3.24340	0.30832	17.2920	56.0849
31	3.37313	0.29646	17.5885	59.3283
32	3.50806	0.28506	17.8736	62.7015
33	3.64838	0.27409	18.1476	66.2095
34	3.79432	0.26355	18.4112	69.8579
35	3.94609	0.25342	18.6646	73.6522
36	4.10393	0.24367	18.9083	77.5983
37	4.26809	0.23430	19.1426	81.7022
38	4.43881	0.22529	19.3679	85.9703
39	4.61637	0.21662	19.5845	90.4091
40	4.80102	0.20829	19.7928	95.0255
41	4.99306	0.20028	19.9931	99.8265
42	5.19278	0.19257	20.1856	104.8196
43	5.40050	0.18517	20.3708	110.0124
44	5.61652	0.17805	20.5488	115.4129
45	5.84118	0.17120	20.7200	121.0294
46	6.07482	0.16461	20.8847	126.8706
47	6.31782	0.15828	21.0429	132.9454
48	6.57053	0.15219	21.1951	139.2632
49	6.83335	0.14634	21.3415	145.8337
50	7.10668	0.14071	21.4822	152.6671
60	10.51963	0.09506	22.6235	237.9907
70	15.57162	0.06422	23.3945	364.2905
80	23.04980	0.04338	23.9154	551.2450
90	34.11933	0.02931	24.2673	827.9833
100	50.50495	0.01980	24.5050	1237.6237

4.5 per cent

i = 0.045000
$i^{(2)}$ = 0.044505
$i^{(4)}$ = 0.044260
$i^{(12)}$ = 0.044098
δ = 0.044017
$(1+i)^{1/2}$ = 1.022252
$(1+i)^{1/4}$ = 1.011065
$(1+i)^{1/12}$ = 1.003675
v = 0.956938
$v^{1/2}$ = 0.978232
$v^{1/4}$ = 0.989056
$v^{1/12}$ = 0.996339
d = 0.043062
$d^{(2)}$ = 0.043536
$d^{(4)}$ = 0.043776
$d^{(12)}$ = 0.043936
$i/i^{(2)}$ = 1.011126
$i/i^{(4)}$ = 1.016720
$i/i^{(12)}$ = 1.020461
i/δ = 1.022335
$i/d^{(2)}$ = 1.033626
$i/d^{(4)}$ = 1.027970
$i/d^{(12)}$ = 1.024211

n	$(1+i)^n$	v^n	$a_{\overline{n}\rceil}$	$s_{\overline{n}\rceil}$
1	1.04500	0.95694	0.9569	1.0000
2	1.09203	0.91573	1.8727	2.0450
3	1.14117	0.87630	2.7490	3.1370
4	1.19252	0.83856	3.5875	4.2782
5	1.24618	0.80245	4.3900	5.4707
6	1.30226	0.76790	5.1579	6.7169
7	1.36086	0.73483	5.8927	8.0192
8	1.42210	0.70319	6.5959	9.3800
9	1.48610	0.67290	7.2688	10.8021
10	1.55297	0.64393	7.9127	12.2882
11	1.62285	0.61620	8.5289	13.8412
12	1.69588	0.58966	9.1186	15.4640
13	1.77220	0.56427	9.6829	17.1599
14	1.85194	0.53997	10.2228	18.9321
15	1.93528	0.51672	10.7395	20.7841
16	2.02237	0.49447	11.2340	22.7193
17	2.11338	0.47318	11.7072	24.7417
18	2.20848	0.45280	12.1600	26.8551
19	2.30786	0.43330	12.5933	29.0636
20	2.41171	0.41464	13.0079	31.3714
21	2.52024	0.39679	13.4047	33.7831
22	2.63365	0.37970	13.7844	36.3034
23	2.75217	0.36335	14.1478	38.9370
24	2.87601	0.34770	14.4955	41.6892
25	3.00543	0.33273	14.8282	44.5652
26	3.14068	0.31840	15.1466	47.5706
27	3.28201	0.30469	15.4513	50.7113
28	3.42970	0.29157	15.7429	53.9933
29	3.58404	0.27902	16.0219	57.4230
30	3.74532	0.26700	16.2889	61.0071
31	3.91386	0.25550	16.5444	64.7524
32	4.08998	0.24450	16.7889	68.6662
33	4.27403	0.23397	17.0229	72.7562
34	4.46636	0.22390	17.2468	77.0303
35	4.66735	0.21425	17.4610	81.4966
36	4.87738	0.20503	17.6660	86.1640
37	5.09686	0.19620	17.8622	91.0413
38	5.32622	0.18775	18.0500	96.1382
39	5.56590	0.17967	18.2297	101.4644
40	5.81636	0.17193	18.4016	107.0303
41	6.07810	0.16453	18.5661	112.8467
42	6.35162	0.15744	18.7235	118.9248
43	6.63744	0.15066	18.8742	125.2764
44	6.93612	0.14417	19.0184	131.9138
45	7.24825	0.13796	19.1563	138.8500
46	7.57442	0.13202	19.2884	146.0982
47	7.91527	0.12634	19.4147	153.6726
48	8.27146	0.12090	19.5356	161.5879
49	8.64367	0.11569	19.6513	169.8594
50	9.03264	0.11071	19.7620	178.5030
60	14.02741	0.07129	20.6380	289.4980
70	21.78414	0.04590	21.2021	461.8697
80	33.83010	0.02956	21.5653	729.5577
90	52.53711	0.01903	21.7992	1145.2690
100	81.58852	0.01226	21.9499	1790.8560

5 per cent

$i = 0.050000$
$i^{(2)} = 0.049390$
$i^{(4)} = 0.049089$
$i^{(12)} = 0.048889$
$\delta = 0.048790$
$(1+i)^{1/2} = 1.024695$
$(1+i)^{1/4} = 1.012272$
$(1+i)^{1/12} = 1.004074$
$v = 0.952381$
$v^{1/2} = 0.975900$
$v^{1/4} = 0.987877$
$v^{1/12} = 0.995942$
$d = 0.047619$
$d^{(2)} = 0.048200$
$d^{(4)} = 0.048494$
$d^{(12)} = 0.048691$
$i/i^{(2)} = 1.012348$
$i/i^{(4)} = 1.018559$
$i/i^{(12)} = 1.022715$
$i/\delta = 1.024797$
$i/d^{(2)} = 1.037348$
$i/d^{(4)} = 1.031059$
$i/d^{(12)} = 1.026881$

n	$(1+i)^n$	v^n	$a_{\overline{n\vert}}$	$s_{\overline{n\vert}}$
1	1.05000	0.95238	0.9524	1.0000
2	1.10250	0.90703	1.8594	2.0500
3	1.15763	0.86384	2.7232	3.1525
4	1.21551	0.82270	3.5460	4.3101
5	1.27628	0.78353	4.3295	5.5256
6	1.34010	0.74622	5.0757	6.8019
7	1.40710	0.71068	5.7864	8.1420
8	1.47746	0.67684	6.4632	9.5491
9	1.55133	0.64461	7.1078	11.0266
10	1.62889	0.61391	7.7217	12.5779
11	1.71034	0.58468	8.3064	14.2068
12	1.79586	0.55684	8.8633	15.9171
13	1.88565	0.53032	9.3936	17.7130
14	1.97993	0.50507	9.8986	19.5986
15	2.07893	0.48102	10.3797	21.5786
16	2.18287	0.45811	10.8378	23.6575
17	2.29202	0.43630	11.2741	25.8404
18	2.40662	0.41552	11.6896	28.1324
19	2.52695	0.39573	12.0853	30.5390
20	2.65330	0.37689	12.4622	33.0660
21	2.78596	0.35894	12.8212	35.7193
22	2.92526	0.34185	13.1630	38.5052
23	3.07152	0.32557	13.4886	41.4305
24	3.22510	0.31007	13.7986	44.5020
25	3.38635	0.29530	14.0939	47.7271
26	3.55567	0.28124	14.3752	51.1135
27	3.73346	0.26785	14.6430	54.6691
28	3.92013	0.25509	14.8981	58.4026
29	4.11614	0.24295	15.1411	62.3227
30	4.32194	0.23138	15.3725	66.4388
31	4.53804	0.22036	15.5928	70.7608
32	4.76494	0.20987	15.8027	75.2988
33	5.00319	0.19987	16.0025	80.0638
34	5.25335	0.19035	16.1929	85.0670
35	5.51602	0.18129	16.3742	90.3203
36	5.79182	0.17266	16.5469	95.8363
37	6.08141	0.16444	16.7113	101.6281
38	6.38548	0.15661	16.8679	107.7095
39	6.70475	0.14915	17.0170	114.0950
40	7.03999	0.14205	17.1591	120.7998
41	7.39199	0.13528	17.2944	127.8398
42	7.76159	0.12884	17.4232	135.2318
43	8.14967	0.12270	17.5459	142.9933
44	8.55715	0.11686	17.6628	151.1430
45	8.98501	0.11130	17.7741	159.7002
46	9.43426	0.10600	17.8801	168.6852
47	9.90597	0.10095	17.9810	178.1194
48	10.40127	0.09614	18.0772	188.0254
49	10.92133	0.09156	18.1687	198.4267
50	11.46740	0.08720	18.2559	209.3480
60	18.67919	0.05354	18.9293	353.5837
70	30.42643	0.03287	19.3427	588.5285
80	49.56144	0.02018	19.5965	971.2288
90	80.73037	0.01239	19.7523	1594.6073
100	131.50126	0.00760	19.8479	2610.0252

5.5 per cent

i	= 0.055000
$i^{(2)}$	= 0.054264
$i^{(4)}$	= 0.053901
$i^{(12)}$	= 0.053660
δ	= 0.053541
$(1+i)^{1/2}$	= 1.027132
$(1+i)^{1/4}$	= 1.013475
$(1+i)^{1/12}$	= 1.004472
v	= 0.947867
$v^{1/2}$	= 0.973585
$v^{1/4}$	= 0.986704
$v^{1/12}$	= 0.995548
d	= 0.052133
$d^{(2)}$	= 0.052830
$d^{(4)}$	= 0.053184
$d^{(12)}$	= 0.053422
$i/i^{(2)}$	= 1.013566
$i/i^{(4)}$	= 1.020395
$i/i^{(12)}$	= 1.024965
i/δ	= 1.027255
$i/d^{(2)}$	= 1.041066
$i/d^{(4)}$	= 1.034145
$i/d^{(12)}$	= 1.029548

n	$(1+i)^n$	v^n	$a_{\overline{n}\rceil}$	$s_{\overline{n}\rceil}$
1	1.05500	0.94787	0.9479	1.0000
2	1.11303	0.89845	1.8463	2.0550
3	1.17424	0.85161	2.6979	3.1680
4	1.23882	0.80722	3.5052	4.3423
5	1.30696	0.76513	4.2703	5.5811
6	1.37884	0.72525	4.9955	6.8881
7	1.45468	0.68744	5.6830	8.2669
8	1.53469	0.65160	6.3346	9.7216
9	1.61909	0.61763	6.9522	11.2563
10	1.70814	0.58543	7.5376	12.8754
11	1.80209	0.55491	8.0925	14.5835
12	1.90121	0.52598	8.6185	16.3856
13	2.00577	0.49856	9.1171	18.2868
14	2.11609	0.47257	9.5896	20.2926
15	2.23248	0.44793	10.0376	22.4087
16	2.35526	0.42458	10.4622	24.6411
17	2.48480	0.40245	10.8646	26.9964
18	2.62147	0.38147	11.2461	29.4812
19	2.76565	0.36158	11.6077	32.1027
20	2.91776	0.34273	11.9504	34.8683
21	3.07823	0.32486	12.2752	37.7861
22	3.24754	0.30793	12.5832	40.8643
23	3.42615	0.29187	12.8750	44.1118
24	3.61459	0.27666	13.1517	47.5380
25	3.81339	0.26223	13.4139	51.1526
26	4.02313	0.24856	13.6625	54.9660
27	4.24440	0.23560	13.8981	58.9891
28	4.47784	0.22332	14.1214	63.2335
29	4.72412	0.21168	14.3331	67.7114
30	4.98395	0.20064	14.5337	72.4355
31	5.25807	0.19018	14.7239	77.4194
32	5.54726	0.18027	14.9042	82.6775
33	5.85236	0.17087	15.0751	88.2248
34	6.17424	0.16196	15.2370	94.0771
35	6.51383	0.15352	15.3906	100.2514
36	6.87209	0.14552	15.5361	106.7652
37	7.25005	0.13793	15.6740	113.6373
38	7.64880	0.13074	15.8047	120.8873
39	8.06949	0.12392	15.9287	128.5361
40	8.51331	0.11746	16.0461	136.6056
41	8.98154	0.11134	16.1575	145.1189
42	9.47553	0.10554	16.2630	154.1005
43	9.99668	0.10003	16.3630	163.5760
44	10.54650	0.09482	16.4579	173.5727
45	11.12655	0.08988	16.5477	184.1192
46	11.73851	0.08519	16.6329	195.2457
47	12.38413	0.08075	16.7137	206.9842
48	13.06526	0.07654	16.7902	219.3684
49	13.78385	0.07255	16.8628	232.4336
50	14.54196	0.06877	16.9315	246.2175
60	24.83977	0.04026	17.4499	433.4504
70	42.42992	0.02357	17.7533	753.2712
80	72.47643	0.01380	17.9310	1299.5714
90	123.80021	0.00808	18.0350	2232.7310
100	211.46864	0.00473	18.0958	3826.7025

6 per cent

$i = 0.060000$
$i^{(2)} = 0.059126$
$i^{(4)} = 0.058695$
$i^{(12)} = 0.058411$
$\delta = 0.058269$
$(1 + i)^{1/2} = 1.029563$
$(1 + i)^{1/4} = 1.014674$
$(1 + i)^{1/12} = 1.004868$
$v = 0.943396$
$v^{1/2} = 0.971286$
$v^{1/4} = 0.985538$
$v^{1/12} = 0.995156$
$d = 0.056604$
$d^{(2)} = 0.057428$
$d^{(4)} = 0.057847$
$d^{(12)} = 0.058128$
$i/i^{(2)} = 1.014782$
$i/i^{(4)} = 1.022227$
$i/i^{(12)} = 1.027211$
$i/\delta = 1.029709$
$i/d^{(2)} = 1.044782$
$i/d^{(4)} = 1.037227$
$i/d^{(12)} = 1.032211$

n	$(1+i)^n$	v^n	$a_{\overline{n}\rceil}$	$s_{\overline{n}\rceil}$
1	1.06000	0.94340	0.9434	1.0000
2	1.12360	0.89000	1.8334	2.0600
3	1.19102	0.83962	2.6730	3.1836
4	1.26248	0.79209	3.4651	4.3746
5	1.33823	0.74726	4.2124	5.6371
6	1.41852	0.70496	4.9173	6.9753
7	1.50363	0.66506	5.5824	8.3938
8	1.59385	0.62741	6.2098	9.8975
9	1.68948	0.59190	6.8017	11.4913
10	1.79085	0.55839	7.3601	13.1808
11	1.89830	0.52679	7.8869	14.9716
12	2.01220	0.49697	8.3838	16.8699
13	2.13293	0.46884	8.8527	18.8821
14	2.26090	0.44230	9.2950	21.0151
15	2.39656	0.41727	9.7122	23.2760
16	2.54035	0.39365	10.1059	25.6725
17	2.69277	0.37136	10.4773	28.2129
18	2.85434	0.35034	10.8276	30.9057
19	3.02560	0.33051	11.1581	33.7600
20	3.20714	0.31180	11.4699	36.7856
21	3.39956	0.29416	11.7641	39.9927
22	3.60354	0.27751	12.0416	43.3923
23	3.81975	0.26180	12.3034	46.9958
24	4.04893	0.24698	12.5504	50.8156
25	4.29187	0.23300	12.7834	54.8645
26	4.54938	0.21981	13.0032	59.1564
27	4.82235	0.20737	13.2105	63.7058
28	5.11169	0.19563	13.4062	68.5281
29	5.41839	0.18456	13.5907	73.6398
30	5.74349	0.17411	13.7648	79.0582
31	6.08810	0.16425	13.9291	84.8017
32	6.45339	0.15496	14.0840	90.8898
33	6.84059	0.14619	14.2302	97.3432
34	7.25103	0.13791	14.3681	104.1838
35	7.68609	0.13011	14.4982	111.4348
36	8.14725	0.12274	14.6210	119.1209
37	8.63609	0.11579	14.7368	127.2681
38	9.15425	0.10924	14.8460	135.9042
39	9.70351	0.10306	14.9491	145.0585
40	10.28572	0.09722	15.0463	154.7620
41	10.90286	0.09172	15.1380	165.0477
42	11.55703	0.08653	15.2245	175.9505
43	12.25045	0.08163	15.3062	187.5076
44	12.98548	0.07701	15.3832	199.7580
45	13.76461	0.07265	15.4558	212.7435
46	14.59049	0.06854	15.5244	226.5081
47	15.46592	0.06466	15.5890	241.0986
48	16.39387	0.06100	15.6500	256.5645
49	17.37750	0.05755	15.7076	272.9584
50	18.42015	0.05429	15.7619	290.3359
60	32.98769	0.03031	16.1614	533.1282
70	59.07593	0.01693	16.3845	967.9322
80	105.79599	0.00945	16.5091	1746.5999
90	189.46451	0.00528	16.5787	3141.0752
100	339.30208	0.00295	16.6175	5638.3681

7 per cent

i =	0.070000
$i^{(2)}$ =	0.068816
$i^{(4)}$ =	0.068234
$i^{(12)}$ =	0.067850
δ =	0.067659
$(1+i)^{1/2}$ =	1.034408
$(1+i)^{1/4}$ =	1.017059
$(1+i)^{1/12}$ =	1.005654
v =	0.934579
$v^{1/2}$ =	0.966736
$v^{1/4}$ =	0.983228
$v^{1/12}$ =	0.994378
d =	0.065421
$d^{(2)}$ =	0.066527
$d^{(4)}$ =	0.067090
$d^{(12)}$ =	0.067468
$i/i^{(2)}$ =	1.017204
$i/i^{(4)}$ =	1.025880
$i/i^{(12)}$ =	1.031691
i/δ =	1.034605
$i/d^{(2)}$ =	1.052204
$i/d^{(4)}$ =	1.043380
$i/d^{(12)}$ =	1.037525

n	$(1+i)^n$	v^n	$a_{\overline{n}\rceil}$	$s_{\overline{n}\rceil}$
1	1.07000	0.93458	0.9346	1.0000
2	1.14490	0.87344	1.8080	2.0700
3	1.22504	0.81630	2.6243	3.2149
4	1.31080	0.76290	3.3872	4.4399
5	1.40255	0.71299	4.1002	5.7507
6	1.50073	0.66634	4.7665	7.1533
7	1.60578	0.62275	5.3893	8.6540
8	1.71819	0.58201	5.9713	10.2598
9	1.83846	0.54393	6.5152	11.9780
10	1.96715	0.50835	7.0236	13.8164
11	2.10485	0.47509	7.4987	15.7836
12	2.25219	0.44401	7.9427	17.8885
13	2.40985	0.41496	8.3577	20.1406
14	2.57853	0.38782	8.7455	22.5505
15	2.75903	0.36245	9.1079	25.1290
16	2.95216	0.33873	9.4466	27.8881
17	3.15882	0.31657	9.7632	30.8402
18	3.37993	0.29586	10.0591	33.9990
19	3.61653	0.27651	10.3356	37.3790
20	3.86968	0.25842	10.5940	40.9955
21	4.14056	0.24151	10.8355	44.8652
22	4.43040	0.22571	11.0612	49.0057
23	4.74053	0.21095	11.2722	53.4361
24	5.07237	0.19715	11.4693	58.1767
25	5.42743	0.18425	11.6536	63.2490
26	5.80735	0.17220	11.8258	68.6765
27	6.21387	0.16093	11.9867	74.4838
28	6.64884	0.15040	12.1371	80.6977
29	7.11426	0.14056	12.2777	87.3465
30	7.61226	0.13137	12.4090	94.4608
31	8.14511	0.12277	12.5318	102.0730
32	8.71527	0.11474	12.6466	110.2182
33	9.32534	0.10723	12.7538	118.9334
34	9.97811	0.10022	12.8540	128.2588
35	10.67658	0.09366	12.9477	138.2369
36	11.42394	0.08754	13.0352	148.9135
37	12.22362	0.08181	13.1170	160.3374
38	13.07927	0.07646	13.1935	172.5610
39	13.99482	0.07146	13.2649	185.6403
40	14.97446	0.06678	13.3317	199.6351
41	16.02267	0.06241	13.3941	214.6096
42	17.14426	0.05833	13.4524	230.6322
43	18.34435	0.05451	13.5070	247.7765
44	19.62846	0.05095	13.5579	266.1209
45	21.00245	0.04761	13.6055	285.7493
46	22.47262	0.04450	13.6500	306.7518
47	24.04571	0.04159	13.6916	329.2244
48	25.72891	0.03887	13.7305	353.2701
49	27.52993	0.03632	13.7668	378.9990
50	29.45703	0.03395	13.8007	406.5289
60	57.94643	0.01726	14.0392	813.5204
70	113.98939	0.00877	14.1604	1614.1342
80	224.23439	0.00446	14.2220	3189.0627
90	441.10298	0.00227	14.2533	6287.1854
100	867.71633	0.00115	14.2693	12381.6618

8 per cent

i =	0.080000
$i^{(2)}$ =	0.078461
$i^{(4)}$ =	0.077706
$i^{(12)}$ =	0.077208
δ =	0.076961
$(1+i)^{1/2}$ =	1.039230
$(1+i)^{1/4}$ =	1.019427
$(1+i)^{1/12}$ =	1.006434
v =	0.925926
$v^{1/2}$ =	0.962250
$v^{1/4}$ =	0.980944
$v^{1/12}$ =	0.993607
d =	0.074074
$d^{(2)}$ =	0.075499
$d^{(4)}$ =	0.076225
$d^{(12)}$ =	0.076715
$i/i^{(2)}$ =	1.019615
$i/i^{(4)}$ =	1.029519
$i/i^{(12)}$ =	1.036157
i/δ =	1.039487
$i/d^{(2)}$ =	1.059615
$i/d^{(4)}$ =	1.049519
$i/d^{(12)}$ =	1.042824

n	$(1+i)^n$	v^n	$a_{\overline{n}\vert}$	$s_{\overline{n}\vert}$
1	1.08000	0.92593	0.9259	1.0000
2	1.16640	0.85734	1.7833	2.0800
3	1.25971	0.79383	2.5771	3.2464
4	1.36049	0.73503	3.3121	4.5061
5	1.46933	0.68058	3.9927	5.8666
6	1.58687	0.63017	4.6229	7.3359
7	1.71382	0.58349	5.2064	8.9228
8	1.85093	0.54027	5.7466	10.6366
9	1.99900	0.50025	6.2469	12.4876
10	2.15892	0.46319	6.7101	14.4866
11	2.33164	0.42888	7.1390	16.6455
12	2.51817	0.39711	7.5361	18.9771
13	2.71962	0.36770	7.9038	21.4953
14	2.93719	0.34046	8.2442	24.2149
15	3.17217	0.31524	8.5595	27.1521
16	3.42594	0.29189	8.8514	30.3243
17	3.70002	0.27027	9.1216	33.7502
18	3.99602	0.25025	9.3719	37.4502
19	4.31570	0.23171	9.6036	41.4463
20	4.66096	0.21455	9.8181	45.7620
21	5.03383	0.19866	10.0168	50.4229
22	5.43654	0.18394	10.2007	55.4568
23	5.87146	0.17032	10.3711	60.8933
24	6.34118	0.15770	10.5288	66.7648
25	6.84848	0.14602	10.6748	73.1059
26	7.39635	0.13520	10.8100	79.9544
27	7.98806	0.12519	10.9352	87.3508
28	8.62711	0.11591	11.0511	95.3388
29	9.31727	0.10733	11.1584	103.9659
30	10.06266	0.09938	11.2578	113.2832
31	10.86767	0.09202	11.3498	123.3459
32	11.73708	0.08520	11.4350	134.2135
33	12.67605	0.07889	11.5139	145.9506
34	13.69013	0.07305	11.5869	158.6267
35	14.78534	0.06763	11.6546	172.3168
36	15.96817	0.06262	11.7172	187.1021
37	17.24563	0.05799	11.7752	203.0703
38	18.62528	0.05369	11.8289	220.3159
39	20.11530	0.04971	11.8786	238.9412
40	21.72452	0.04603	11.9246	259.0565
41	23.46248	0.04262	11.9672	280.7810
42	25.33948	0.03946	12.0067	304.2435
43	27.36664	0.03654	12.0432	329.5830
44	29.55597	0.03383	12.0771	356.9496
45	31.92045	0.03133	12.1084	386.5056
46	34.47409	0.02901	12.1374	418.4261
47	37.23201	0.02686	12.1643	452.9002
48	40.21057	0.02487	12.1891	490.1322
49	43.42742	0.02303	12.2122	530.3427
50	46.90161	0.02132	12.2335	573.7702
60	101.25706	0.00988	12.3766	1253.2133
70	218.60641	0.00457	12.4428	2720.0801
80	471.95483	0.00212	12.4735	5886.9354
90	1018.91509	0.00098	12.4877	12723.9386
100	2199.76126	0.00045	12.4943	27484.5157

9 per cent

$i = 0.090000$
$i^{(2)} = 0.088061$
$i^{(4)} = 0.087113$
$i^{(12)} = 0.086488$
$\delta = 0.086178$
$(1+i)^{1/2} = 1.044031$
$(1+i)^{1/4} = 1.021778$
$(1+i)^{1/12} = 1.007207$
$v = 0.917431$
$v^{1/2} = 0.957826$
$v^{1/4} = 0.978686$
$v^{1/12} = 0.992844$
$d = 0.082569$
$d^{(2)} = 0.084347$
$d^{(4)} = 0.085256$
$d^{(12)} = 0.085869$
$i/i^{(2)} = 1.022015$
$i/i^{(4)} = 1.033144$
$i/i^{(12)} = 1.040608$
$i/\delta = 1.044354$
$i/d^{(2)} = 1.067015$
$i/d^{(4)} = 1.055644$
$i/d^{(12)} = 1.048108$

n	$(1+i)^n$	v^n	$a_{\overline{n}\rceil}$	$s_{\overline{n}\rceil}$
1	1.09000	0.91743	0.9174	1.0000
2	1.18810	0.84168	1.7591	2.0900
3	1.29503	0.77218	2.5313	3.2781
4	1.41158	0.70843	3.2397	4.5731
5	1.53862	0.64993	3.8897	5.9847
6	1.67710	0.59627	4.4859	7.5233
7	1.82804	0.54703	5.0330	9.2004
8	1.99256	0.50187	5.5348	11.0285
9	2.17189	0.46043	5.9952	13.0210
10	2.36736	0.42241	6.4177	15.1929
11	2.58043	0.38753	6.8052	17.5603
12	2.81266	0.35553	7.1607	20.1407
13	3.06580	0.32618	7.4869	22.9534
14	3.34173	0.29925	7.7862	26.0192
15	3.64248	0.27454	8.0607	29.3609
16	3.97031	0.25187	8.3126	33.0034
17	4.32763	0.23107	8.5436	36.9737
18	4.71712	0.21199	8.7556	41.3013
19	5.14166	0.19449	8.9501	46.0185
20	5.60441	0.17843	9.1285	51.1601
21	6.10881	0.16370	9.2922	56.7645
22	6.65860	0.15018	9.4424	62.8733
23	7.25787	0.13778	9.5802	69.5319
24	7.91108	0.12640	9.7066	76.7898
25	8.62308	0.11597	9.8226	84.7009
26	9.39916	0.10639	9.9290	93.3240
27	10.24508	0.09761	10.0266	102.7231
28	11.16714	0.08955	10.1161	112.9682
29	12.17218	0.08215	10.1983	124.1354
30	13.26768	0.07537	10.2737	136.3075
31	14.46177	0.06915	10.3428	149.5752
32	15.76333	0.06344	10.4062	164.0370
33	17.18203	0.05820	10.4644	179.8003
34	18.72841	0.05339	10.5178	196.9823
35	20.41397	0.04899	10.5668	215.7108
36	22.25123	0.04494	10.6118	236.1247
37	24.25384	0.04123	10.6530	258.3759
38	26.43668	0.03783	10.6908	282.6298
39	28.81598	0.03470	10.7255	309.0665
40	31.40942	0.03184	10.7574	337.8824
41	34.23627	0.02921	10.7866	369.2919
42	37.31753	0.02680	10.8134	403.5281
43	40.67611	0.02458	10.8380	440.8457
44	44.33696	0.02255	10.8605	481.5218
45	48.32729	0.02069	10.8812	525.8587
46	52.67674	0.01898	10.9002	574.1860
47	57.41765	0.01742	10.9176	626.8628
48	62.58524	0.01598	10.9336	684.2804
49	68.21791	0.01466	10.9482	746.8656
50	74.35752	0.01345	10.9617	815.0836
60	176.03129	0.00568	11.0480	1944.7921
70	416.73009	0.00240	11.0844	4619.2232
80	986.55167	0.00101	11.0998	10950.5741
90	2335.52658	0.00043	11.1064	25939.1842
100	5529.04079	0.00018	11.1091	61422.6755

10 per cent

$i = 0.100000$
$i^{(2)} = 0.097618$
$i^{(4)} = 0.096455$
$i^{(12)} = 0.095690$
$\delta = 0.095310$
$(1+i)^{1/2} = 1.048809$
$(1+i)^{1/4} = 1.024114$
$(1+i)^{1/12} = 1.007974$
$v = 0.909091$
$v^{1/2} = 0.953463$
$v^{1/4} = 0.976454$
$v^{1/12} = 0.992089$
$d = 0.090909$
$d^{(2)} = 0.093075$
$d^{(4)} = 0.094184$
$d^{(12)} = 0.094933$
$i/i^{(2)} = 1.024404$
$i/i^{(4)} = 1.036756$
$i/i^{(12)} = 1.045045$
$i/\delta = 1.049206$
$i/d^{(2)} = 1.074404$
$i/d^{(4)} = 1.061756$
$i/d^{(12)} = 1.053378$

n	$(1+i)^n$	v^n	$a_{\overline{n}\rceil}$	$s_{\overline{n}\rceil}$
1	1.10000	0.90909	0.9091	1.0000
2	1.21000	0.82645	1.7355	2.1000
3	1.33100	0.75131	2.4869	3.3100
4	1.46410	0.68301	3.1699	4.6410
5	1.61051	0.62092	3.7908	6.1051
6	1.77156	0.56447	4.3553	7.7156
7	1.94872	0.51316	4.8684	9.4872
8	2.14359	0.46651	5.3349	11.4359
9	2.35795	0.42410	5.7590	13.5795
10	2.59374	0.38554	6.1446	15.9374
11	2.85312	0.35049	6.4951	18.5312
12	3.13843	0.31863	6.8137	21.3843
13	3.45227	0.28966	7.1034	24.5227
14	3.79750	0.26333	7.3667	27.9750
15	4.17725	0.23939	7.6061	31.7725
16	4.59497	0.21763	7.8237	35.9497
17	5.05447	0.19784	8.0216	40.5447
18	5.55992	0.17986	8.2014	45.5992
19	6.11591	0.16351	8.3649	51.1591
20	6.72750	0.14864	8.5136	57.2750
21	7.40025	0.13513	8.6487	64.0025
22	8.14027	0.12285	8.7715	71.4027
23	8.95430	0.11168	8.8832	79.5430
24	9.84973	0.10153	8.9847	88.4973
25	10.83471	0.09230	9.0770	98.3471
26	11.91818	0.08391	9.1609	109.1818
27	13.10999	0.07628	9.2372	121.0999
28	14.42099	0.06934	9.3066	134.2099
29	15.86309	0.06304	9.3696	148.6309
30	17.44940	0.05731	9.4269	164.4940
31	19.19434	0.05210	9.4790	181.9434
32	21.11378	0.04736	9.5264	201.1378
33	23.22515	0.04306	9.5694	222.2515
34	25.54767	0.03914	9.6086	245.4767
35	28.10244	0.03558	9.6442	271.0244
36	30.91268	0.03235	9.6765	299.1268
37	34.00395	0.02941	9.7059	330.0395
38	37.40434	0.02673	9.7327	364.0434
39	41.14478	0.02430	9.7570	401.4478
40	45.25926	0.02209	9.7791	442.5926
41	49.78518	0.02009	9.7991	487.8518
42	54.76370	0.01826	9.8174	537.6370
43	60.24007	0.01660	9.8340	592.4007
44	66.26408	0.01509	9.8491	652.6408
45	72.89048	0.01372	9.8628	718.9048
46	80.17953	0.01247	9.8753	791.7953
47	88.19749	0.01134	9.8866	871.9749
48	97.01723	0.01031	9.8969	960.1723
49	106.71896	0.00937	9.9063	1057.1896
50	117.39085	0.00852	9.9148	1163.9085
60	304.48164	0.00328	9.9672	3034.8164
70	789.74696	0.00127	9.9873	7887.4696
80	2048.40021	0.00049	9.9951	20474.0021
90	5313.02261	0.00019	9.9981	53120.2261
100	13780.61234	0.00007	9.9993	137796.1234

12 per cent

$i = 0.120000$
$i^{(2)} = 0.116601$
$i^{(4)} = 0.114949$
$i^{(12)} = 0.113866$
$\delta = 0.113329$
$(1+i)^{1/2} = 1.058301$
$(1+i)^{1/4} = 1.028737$
$(1+i)^{1/12} = 1.009489$
$v = 0.892857$
$v^{1/2} = 0.944911$
$v^{1/4} = 0.972065$
$v^{1/12} = 0.990600$
$d = 0.107143$
$d^{(2)} = 0.110178$
$d^{(4)} = 0.111738$
$d^{(12)} = 0.112795$
$i/i^{(2)} = 1.029150$
$i/i^{(4)} = 1.043938$
$i/i^{(12)} = 1.053875$
$i/\delta = 1.058867$
$i/d^{(2)} = 1.089150$
$i/d^{(4)} = 1.073938$
$i/d^{(12)} = 1.063875$

n	$(1+i)^n$	v^n	$a_{\overline{n}\vert}$	$s_{\overline{n}\vert}$
1	1.12000	0.89286	0.8929	1.0000
2	1.25440	0.79719	1.6901	2.1200
3	1.40493	0.71178	2.4018	3.3744
4	1.57352	0.63552	3.0373	4.7793
5	1.76234	0.56743	3.6048	6.3528
6	1.97382	0.50663	4.1114	8.1152
7	2.21068	0.45235	4.5638	10.0890
8	2.47596	0.40388	4.9676	12.2997
9	2.77308	0.36061	5.3282	14.7757
10	3.10585	0.32197	5.6502	17.5487
11	3.47855	0.28748	5.9377	20.6546
12	3.89598	0.25668	6.1944	24.1331
13	4.36349	0.22917	6.4235	28.0291
14	4.88711	0.20462	6.6282	32.3926
15	5.47357	0.18270	6.8109	37.2797
16	6.13039	0.16312	6.9740	42.7533
17	6.86604	0.14564	7.1196	48.8837
18	7.68997	0.13004	7.2497	55.7497
19	8.61276	0.11611	7.3658	63.4397
20	9.64629	0.10367	7.4694	72.0524
21	10.80385	0.09256	7.5620	81.6987
22	12.10031	0.08264	7.6446	92.5026
23	13.55235	0.07379	7.7184	104.6029
24	15.17863	0.06588	7.7843	118.1552
25	17.00006	0.05882	7.8431	133.3339
26	19.04007	0.05252	7.8957	150.3339
27	21.32488	0.04689	7.9426	169.3740
28	23.88387	0.04187	7.9844	190.6989
29	26.74993	0.03738	8.0218	214.5828
30	29.95992	0.03338	8.0552	241.3327
31	33.55511	0.02980	8.0850	271.2926
32	37.58173	0.02661	8.1116	304.8477
33	42.09153	0.02376	8.1354	342.4294
34	47.14252	0.02121	8.1566	384.5210
35	52.79962	0.01894	8.1755	431.6635
36	59.13557	0.01691	8.1924	484.4631
37	66.23184	0.01510	8.2075	543.5987
38	74.17966	0.01348	8.2210	609.8305
39	83.08122	0.01204	8.2330	684.0102
40	93.05097	0.01075	8.2438	767.0914
41	104.21709	0.00960	8.2534	860.1424
42	116.72314	0.00857	8.2619	964.3595
43	130.72991	0.00765	8.2696	1081.0826
44	146.41750	0.00683	8.2764	1211.8125
45	163.98760	0.00610	8.2825	1358.2300
46	183.66612	0.00544	8.2880	1522.2176
47	205.70605	0.00486	8.2928	1705.8838
48	230.39078	0.00434	8.2972	1911.5898
49	258.03767	0.00388	8.3010	2141.9806
50	289.00219	0.00346	8.3045	2400.0182
60	897.59693	0.00111	8.3240	7471.6411
70	2787.79983	0.00036	8.3303	23223.3319
80	8658.48310	0.00012	8.3324	72145.6925
90	26891.93422	0.00004	8.3330	224091.1185
100	83522.26573	0.00001	8.3332	696010.5477

15 per cent

$i = 0.150000$
$i^{(2)} = 0.144761$
$i^{(4)} = 0.142232$
$i^{(12)} = 0.140579$
$\delta = 0.139762$
$(1+i)^{1/2} = 1.072381$
$(1+i)^{1/4} = 1.035558$
$(1+i)^{1/12} = 1.011715$
$v = 0.869565$
$v^{1/2} = 0.932505$
$v^{1/4} = 0.965663$
$v^{1/12} = 0.988421$
$d = 0.130435$
$d^{(2)} = 0.134990$
$d^{(4)} = 0.137348$
$d^{(12)} = 0.138951$
$i/i^{(2)} = 1.036190$
$i/i^{(4)} = 1.054613$
$i/i^{(12)} = 1.067016$
$i/\delta = 1.073254$
$i/d^{(2)} = 1.111190$
$i/d^{(4)} = 1.092113$
$i/d^{(12)} = 1.079516$

n	$(1+i)^n$	v^n	$a_{\overline{n}\rceil}$	$s_{\overline{n}\rceil}$
1	1.15000	0.86957	0.8696	1.0000
2	1.32250	0.75614	1.6257	2.1500
3	1.52088	0.65752	2.2832	3.4725
4	1.74901	0.57175	2.8550	4.9934
5	2.01136	0.49718	3.3522	6.7424
6	2.31306	0.43233	3.7845	8.7537
7	2.66002	0.37594	4.1604	11.0668
8	3.05902	0.32690	4.4873	13.7268
9	3.51788	0.28426	4.7716	16.7858
10	4.04556	0.24718	5.0188	20.3037
11	4.65239	0.21494	5.2337	24.3493
12	5.35025	0.18691	5.4206	29.0017
13	6.15279	0.16253	5.5831	34.3519
14	7.07571	0.14133	5.7245	40.5047
15	8.13706	0.12289	5.8474	47.5804
16	9.35762	0.10686	5.9542	55.7175
17	10.76126	0.09293	6.0472	65.0751
18	12.37545	0.08081	6.1280	75.8364
19	14.23177	0.07027	6.1982	88.2118
20	16.36654	0.06110	6.2593	102.4436
21	18.82152	0.05313	6.3125	118.8101
22	21.64475	0.04620	6.3587	137.6316
23	24.89146	0.04017	6.3988	159.2764
24	28.62518	0.03493	6.4338	184.1678
25	32.91895	0.03038	6.4641	212.7930
26	37.85680	0.02642	6.4906	245.7120
27	43.53531	0.02297	6.5135	283.5688
28	50.06561	0.01997	6.5335	327.1041
29	57.57545	0.01737	6.5509	377.1697
30	66.21177	0.01510	6.5660	434.7451
31	76.14354	0.01313	6.5791	500.9569
32	87.56507	0.01142	6.5905	577.1005
33	100.69983	0.00993	6.6005	664.6655
34	115.80480	0.00864	6.6091	765.3654
35	133.17552	0.00751	6.6166	881.1702
36	153.15185	0.00653	6.6231	1014.3457
37	176.12463	0.00568	6.6288	1167.4975
38	202.54332	0.00494	6.6338	1343.6222
39	232.92482	0.00429	6.6380	1546.1655
40	267.86355	0.00373	6.6418	1779.0903
41	308.04308	0.00325	6.6450	2046.9539
42	354.24954	0.00282	6.6478	2354.9969
43	407.38697	0.00245	6.6503	2709.2465
44	468.49502	0.00213	6.6524	3116.6334
45	538.76927	0.00186	6.6543	3585.1285
46	619.58466	0.00161	6.6559	4123.8977
47	712.52236	0.00140	6.6573	4743.4824
48	819.40071	0.00122	6.6585	5456.0047
49	942.31082	0.00106	6.6596	6275.4055
50	1083.65744	0.00092	6.6605	7217.7163
60	4383.99875	0.00023	6.6651	29219.9916
70	17735.72004	0.00006	6.6663	118231.4669
80	71750.87940	0.00001	6.6666	478332.5293
90	290272.32521	0.00000	6.6666	1935142.1680
100	1174313.45070	0.00000	6.6667	7828749.6713

20 per cent

i	= 0.200000
$i^{(2)}$	= 0.190890
$i^{(4)}$	= 0.186541
$i^{(12)}$	= 0.183714
δ	= 0.182322
$(1+i)^{1/2}$	= 1.095445
$(1+i)^{1/4}$	= 1.046635
$(1+i)^{1/12}$	= 1.015309
v	= 0.833333
$v^{1/2}$	= 0.912871
$v^{1/4}$	= 0.955443
$v^{1/12}$	= 0.984921
d	= 0.166667
$d^{(2)}$	= 0.174258
$d^{(4)}$	= 0.178229
$d^{(12)}$	= 0.180943
$i/i^{(2)}$	= 1.047723
$i/i^{(4)}$	= 1.072153
$i/i^{(12)}$	= 1.088651
i/δ	= 1.096963
$i/d^{(2)}$	= 1.147723
$i/d^{(4)}$	= 1.122153
$i/d^{(12)}$	= 1.105317

n	$(1+i)^n$	v^n	$a_{\overline{n}\rceil}$	$s_{\overline{n}\rceil}$
1	1.20000	0.83333	0.8333	1.0000
2	1.44000	0.69444	1.5278	2.2000
3	1.72800	0.57870	2.1065	3.6400
4	2.07360	0.48225	2.5887	5.3680
5	2.48832	0.40188	2.9906	7.4416
6	2.98598	0.33490	3.3255	9.9299
7	3.58318	0.27908	3.6046	12.9159
8	4.29982	0.23257	3.8372	16.4991
9	5.15978	0.19381	4.0310	20.7989
10	6.19174	0.16151	4.1925	25.9587
11	7.43008	0.13459	4.3271	32.1504
12	8.91610	0.11216	4.4392	39.5805
13	10.69932	0.09346	4.5327	48.4966
14	12.83918	0.07789	4.6106	59.1959
15	15.40702	0.06491	4.6755	72.0351
16	18.48843	0.05409	4.7296	87.4421
17	22.18611	0.04507	4.7746	105.9306
18	26.62333	0.03756	4.8122	128.1167
19	31.94800	0.03130	4.8435	154.7400
20	38.33760	0.02608	4.8696	186.6880
21	46.00512	0.02174	4.8913	225.0256
22	55.20614	0.01811	4.9094	271.0307
23	66.24737	0.01509	4.9245	326.2369
24	79.49685	0.01258	4.9371	392.4842
25	95.39622	0.01048	4.9476	471.9811
26	114.47546	0.00874	4.9563	567.3773
27	137.37055	0.00728	4.9636	681.8528
28	164.84466	0.00607	4.9697	819.2233
29	197.81359	0.00506	4.9747	984.0680
30	237.37631	0.00421	4.9789	1181.8816
31	284.85158	0.00351	4.9824	1419.2579
32	341.82189	0.00293	4.9854	1704.1095
33	410.18627	0.00244	4.9878	2045.9314
34	492.22352	0.00203	4.9898	2456.1176
35	590.66823	0.00169	4.9915	2948.3411
36	708.80187	0.00141	4.9929	3539.0094
37	850.56225	0.00118	4.9941	4247.8112
38	1020.67470	0.00098	4.9951	5098.3735
39	1224.80964	0.00082	4.9959	6119.0482
40	1469.77157	0.00068	4.9966	7343.8578
41	1763.72588	0.00057	4.9972	8813.6294
42	2116.47106	0.00047	4.9976	10577.3553
43	2539.76527	0.00039	4.9980	12693.8263
44	3047.71832	0.00033	4.9984	15233.5916
45	3657.26199	0.00027	4.9986	18281.3099
46	4388.71439	0.00023	4.9989	21938.5719
47	5266.45726	0.00019	4.9991	26327.2863
48	6319.74872	0.00016	4.9992	31593.7436
49	7583.69846	0.00013	4.9993	37913.4923
50	9100.43815	0.00011	4.9995	45497.1908
60	56347.51435	0.00002	4.9999	281732.5718
70	348888.95693	0.00000	5.0000	1744439.7847
80	2160228.46201	0.00000	5.0000	10801137.3101
90	13375565.24893	0.00000	5.0000	66877821.2447
100	82817974.52201	0.00000	5.0000	414089867.6101

25 per cent

i =	0.250000
$i^{(2)}$ =	0.236068
$i^{(4)}$ =	0.229485
$i^{(12)}$ =	0.225231
δ =	0.223144
$(1+i)^{1/2}$ =	1.118034
$(1+i)^{1/4}$ =	1.057371
$(1+i)^{1/12}$ =	1.018769
v =	0.800000
$v^{1/2}$ =	0.894427
$v^{1/4}$ =	0.945742
$v^{1/12}$ =	0.981577
d =	0.200000
$d^{(2)}$ =	0.211146
$d^{(4)}$ =	0.217034
$d^{(12)}$ =	0.221082
$i/i^{(2)}$ =	1.059017
$i/i^{(4)}$ =	1.089396
$i/i^{(12)}$ =	1.109971
i/δ =	1.120355
$i/d^{(2)}$ =	1.184017
$i/d^{(4)}$ =	1.151896
$i/d^{(12)}$ =	1.130804

n	$(1+i)^n$	v^n	$a_{\overline{n}\rceil}$	$s_{\overline{n}\rceil}$
1	1.25000	0.80000	0.8000	1.0000
2	1.56250	0.64000	1.4400	2.2500
3	1.95313	0.51200	1.9520	3.8125
4	2.44141	0.40960	2.3616	5.7656
5	3.05176	0.32768	2.6893	8.2070
6	3.81470	0.26214	2.9514	11.2588
7	4.76837	0.20972	3.1611	15.0735
8	5.96046	0.16777	3.3289	19.8419
9	7.45058	0.13422	3.4631	25.8023
10	9.31323	0.10737	3.5705	33.2529
11	11.64153	0.08590	3.6564	42.5661
12	14.55192	0.06872	3.7251	54.2077
13	18.18989	0.05498	3.7801	68.7596
14	22.73737	0.04398	3.8241	86.9495
15	28.42171	0.03518	3.8593	109.6868
16	35.52714	0.02815	3.8874	138.1085
17	44.40892	0.02252	3.9099	173.6357
18	55.51115	0.01801	3.9279	218.0446
19	69.38894	0.01441	3.9424	273.5558
20	86.73617	0.01153	3.9539	342.9447
21	108.42022	0.00922	3.9631	429.6809
22	135.52527	0.00738	3.9705	538.1011
23	169.40659	0.00590	3.9764	673.6264
24	211.75824	0.00472	3.9811	843.0329
25	264.69780	0.00378	3.9849	1054.7912
26	330.87225	0.00302	3.9879	1319.4890
27	413.59031	0.00242	3.9903	1650.3612
28	516.98788	0.00193	3.9923	2063.9515
29	646.23485	0.00155	3.9938	2580.9394
30	807.79357	0.00124	3.9950	3227.1743
31	1009.74196	0.00099	3.9960	4034.9678
32	1262.17745	0.00079	3.9968	5044.7098
33	1577.72181	0.00063	3.9975	6306.8872
34	1972.15226	0.00051	3.9980	7884.6091
35	2465.19033	0.00041	3.9984	9856.7613
36	3081.48791	0.00032	3.9987	12321.9516
37	3851.85989	0.00026	3.9990	15403.4396
38	4814.82486	0.00021	3.9992	19255.2994
39	6018.53108	0.00017	3.9993	24070.1243
40	7523.16385	0.00013	3.9995	30088.6554
41	9403.95481	0.00011	3.9996	37611.8192
42	11754.94351	0.00009	3.9997	47015.7740
43	14693.67939	0.00007	3.9997	58770.7175
44	18367.09923	0.00005	3.9998	73464.3969
45	22958.87404	0.00004	3.9998	91831.4962
46	28698.59255	0.00003	3.9999	114790.3702
47	35873.24069	0.00003	3.9999	143488.9627
48	44841.55086	0.00002	3.9999	179362.2034
49	56051.93857	0.00002	3.9999	224203.7543
50	70064.92322	0.00001	3.9999	280255.6929
60	652530.44680	0.00000	4.0000	2610117.7872
70	6077163.35729	0.00000	4.0000	24308649.4291
80	56597994.24267	0.00000	4.0000	226391972.9707
90	527109897.16153	0.00000	4.0000	2108439584.6461
100	4909093465.29773	0.00000	4.0000	19636373857.1909

Appendix 2

Illustrative Mortality Table

BASIC FUNCTIONS

Age	ℓ_x	d_x	$1000\, q_x$
0	100000.00	2042.1700	20.4217
1	97957.83	131.5672	1.3431
2	97826.26	119.7100	1.2237
3	97706.55	109.8124	1.1239
4	97596.74	101.7056	1.0421
5	97495.03	95.2526	0.9770
6	97399.78	90.2799	0.9269
7	97309.50	86.6444	0.8904
8	97222.86	84.1950	0.8660
9	97138.66	82.7816	0.8522
10	97055.88	82.2549	0.8475
11	96973.63	82.4664	0.8504
12	96891.16	83.2842	0.8594
13	96807.88	84.5180	0.8730
14	96723.36	86.0611	0.8898
15	96637.30	87.7559	0.9081
16	96549.54	89.6167	0.9282
17	96459.92	91.6592	0.9502
18	96368.27	93.9005	0.9744
19	96274.36	96.3596	1.0009
20	96178.01	99.0569	1.0299
21	96078.95	102.0149	1.0618
22	95976.93	105.2582	1.0967
23	95871.68	108.8135	1.1350
24	95762.86	112.7102	1.1770
25	95650.15	116.9802	1.2330
26	95533.17	121.6585	1.2735
27	95411.51	126.7830	1.3288
28	95284.73	132.3953	1.3895
29	95152.33	138.5406	1.4560
30	95013.79	145.2682	1.5289
31	94868.53	152.6317	1.6089
32	94715.89	160.6896	1.6965
33	94555.20	169.5052	1.7927
34	94385.70	179.1475	1.8980
35	94206.55	189.6914	2.0136
36	94016.86	201.2179	2.1402
37	93815.64	213.8149	2.2791
38	93601.83	227.5775	2.4313
39	93374.25	242.6085	2.5982
40	93131.64	259.0186	2.7812
41	92872.62	276.9271	2.9818
42	92595.70	296.4623	3.2017
43	92299.23	317.7619	3.4427
44	91981.47	340.9730	3.7070
45	91640.50	366.2529	3.9966
46	91274.25	393.7687	4.3141
47	90880.48	423.6978	4.6621
48	90456.78	456.2274	5.0436
49	90000.55	491.5543	5.4617
50	89509.00	529.8844	5.9199
51	88979.11	571.4316	6.4221
52	88407.68	616.4165	6.9724
53	87791.26	665.0646	7.5755
54	87126.20	717.6041	8.2364
55	86408.60	774.2626	8.9605

Source: From Appendix 2A of *Actuarial Mathematics* by Bowers et al., pages 560-569. Copyright 1986 by the Society of Actuaries, Schaumburg, Illinois. Reproduced by permission.

Age	ℓ_x	d_x	$1000\ q_x$
56	85634.33	835.2636	9.7538
57	84799.07	900.8215	10.6230
58	83898.25	971.1358	11.5752
59	82927.11	1046.3843	12.6181
60	81880.73	1126.7146	13.7604
61	80754.01	1212.2343	15.0114
62	79541.78	1302.9994	16.3813
63	78238.78	1399.0010	17.8812
64	76839.78	1500.1504	19.5231
65	75339.63	1606.2618	21.3203
66	73733.37	1717.0334	23.2871
67	72016.33	1832.0273	25.4391
68	70184.31	1950.6476	27.7932
69	68233.66	2072.1177	30.3680
70	66161.54	2195.4578	33.1833
71	63966.08	2319.4639	36.2608
72	61646.62	2442.6884	39.6240
73	59203.93	2563.4258	43.2982
74	56640.51	2679.7050	47.3108
75	53960.80	2789.2905	51.6911
76	51171.51	2889.6965	56.4708
77	48281.81	2978.2164	61.6840
78	45303.60	3051.9717	67.3671
79	42251.62	3107.9833	73.5589
80	39143.64	3143.2679	80.3009
81	36000.37	3154.9603	87.6369
82	32845.41	3140.4624	95.6134
83	29704.95	3097.6146	104.2794
84	26607.34	3024.8830	113.6860
85	23582.45	2921.5530	123.8867
86	20660.90	2787.9129	134.9367
87	17872.99	2625.4088	146.8926
88	15247.58	2436.7474	159.8121
89	12810.83	2225.9244	173.7533
90	10584.91	1998.1533	188.7738
91	8586.75	1759.6818	204.9298
92	6827.07	1517.4869	222.2749
93	5309.58	1278.8606	240.8589
94	4030.72	1050.9136	260.7257
95	2979.81	840.0452	281.9122
96	2139.77	651.4422	304.4456
97	1488.32	488.6776	328.3410
98	999.65	353.4741	353.5993
99	646.17	245.6772	380.2041
100	400.49	163.4494	408.1188
101	237.05	103.6560	437.2837
102	133.39	62.3746	467.6133
103	71.01	35.4358	498.9935
104	35.58	18.9023	531.2793
105	16.68	9.4105	564.2937
106	7.27	4.3438	597.8266
107	2.92	1.8458	631.6360
108	1.08	0.7163	665.4495
109	0.36	0.2517	698.9685
110	0.11	0.0793	731.8742

COMMUTATION FUNCTIONS, D_x, N_x, S_x, $i = 0.06$

Age	D_x	N_x	S_x
0	100000.00	1680095.45	27526802.72
1	92413.05	1580095.45	25846707.27
2	87065.03	1487682.41	24266611.82
3	82036.31	1400617.38	22778929.41
4	77305.76	1318581.07	21378312.03
5	72853.96	1241275.31	20059730.96
6	68663.00	1168421.35	18818455.64
7	64716.38	1099758.35	17650034.29
8	60998.82	1035041.97	16550275.94
9	57496.23	974043.15	15515233.97
10	54195.50	916546.92	14541190.82
11	51084.50	862351.43	13624643.89
12	48151.94	811266.93	12762292.47
13	45387.31	763114.99	11951025.54
14	42780.83	717727.68	11187910.54
15	40323.37	674946.85	10470182.86
16	38006.37	634623.48	9795236.01
17	35821.78	596617.11	9160612.54
18	33762.02	560795.33	8563995.42
19	31819.93	527033.30	8003200.10
20	29988.76	495213.37	7476166.79
21	28262.14	465224.62	6980953.42
22	26634.09	436962.48	6515728.80
23	25098.94	410328.39	6078766.32
24	23651.37	385229.45	5668437.93
25	22286.35	361578.07	5283208.49
26	20999.15	339291.72	4921630.41
27	19785.29	318292.57	4582338.70
28	18640.57	298507.28	4264046.13
29	17561.00	279866.71	3965538.85
30	16542.86	262305.71	3685672.13
31	15582.61	245762.85	3423366.42
32	14676.93	230180.23	3177603.57
33	13822.67	215503.30	2947423.34
34	13016.88	201680.64	2731920.04
35	12256.76	188663.76	2530239.40
36	11539.70	176406.99	2341575.65
37	10863.21	164867.29	2165168.65
38	10224.96	154004.08	2000301.36
39	9622.73	143779.13	1846297.28
40	9054.46	134156.39	1702518.15
41	8518.19	125101.93	1568361.76
42	8012.06	116583.74	1443259.83
43	7534.35	108571.68	1326676.08
44	7083.41	101037.33	1218104.41
45	6657.69	93953.92	1117067.08
46	6255.74	87296.23	1023113.16
47	5876.18	81040.49	935816.93
48	5517.72	75164.31	854776.44
49	5179.14	69646.60	779612.12
50	4859.30	64467.45	709965.53
51	4557.10	59608.16	645498.07
52	4271.55	55051.05	585889.92
53	4001.66	50779.51	530838.86
54	3746.55	46777.85	480059.36
55	3505.37	43031.29	433281.51

Age	D_x	N_x	S_x
56	3277.32	39525.92	390250.22
57	3061.66	36248.59	350724.30
58	2857.67	33186.93	314475.71
59	2664.71	30329.26	281288.77
60	2482.16	27664.55	250959.22
61	2309.44	25182.39	223294.97
62	2146.01	22872.95	198112.58
63	1991.37	20726.94	175239.63
64	1845.06	18735.57	154512.70
65	1706.64	16890.50	135777.13
66	1575.71	15183.86	118886.62
67	1451.90	13608.15	103702.76
68	1334.88	12156.25	90094.61
69	1224.32	10821.37	77938.36
70	1119.94	9597.05	67116.99
71	1021.49	8477.11	57519.94
72	928.72	7455.62	49042.83
73	841.44	6526.90	41587.20
74	759.44	5685.46	35060.31
75	682.56	4926.02	29374.84
76	610.64	4243.47	24448.82
77	543.54	3632.83	20205.36
78	481.14	3089.29	16572.53
79	423.33	2608.14	13483.24
80	369.99	2184.81	10875.10
81	321.02	1814.82	8690.28
82	276.31	1493.80	6875.46
83	235.74	1217.49	5381.66
84	199.21	981.75	4164.17
85	166.57	782.54	3182.42
86	137.67	615.97	2399.88
87	112.35	478.30	1783.90
88	90.42	365.95	1305.60
89	71.67	275.52	939.66
90	55.87	203.85	664.13
91	42.76	147.98	460.28
92	32.07	105.23	312.30
93	23.53	73.16	207.08
94	16.85	49.63	133.92
95	11.75	32.78	84.29
96	7.96	21.02	51.52
97	5.22	13.06	30.49
98	3.31	7.84	17.43
99	2.02	4.53	9.59
100	1.18	2.51	5.07
101	0.66	1.33	2.56
102	0.35	0.67	1.23
103	0.18	0.32	0.56
104	0.08	0.14	0.24
105	0.04	0.06	0.10
106	0.02	0.02	0.04
107	0.01	0.01	0.01
108	0.00	0.00	0.00
109	0.00	0.00	0.00
110	0.00	0.00	0.00

COMMUTATION FUNCTIONS C_x, M_x, R_x, $i = 0.06$

Age	C_x	M_x	R_x
0	1926.5755	4900.2574	121974.5442
1	117.0943	2973.6819	117074.2868
2	100.5108	2856.5876	114100.6049
3	86.9817	2756.0768	111244.0173
4	76.0003	2669.0951	108487.9406
5	67.1494	2593.0948	105818.8455
6	60.0413	2525.9454	103225.7507
7	54.3618	2465.9041	100699.8053
8	49.8349	2411.5424	98233.9012
9	46.2248	2361.7075	95822.3588
10	43.3308	2315.4827	93460.6513
11	40.9833	2272.1519	91145.1686
12	39.0469	2231.1686	88873.0168
13	37.3824	2192.1217	86641.8482
14	35.9013	2154.7393	84449.7264
15	34.5448	2118.8290	82294.9871
16	33.2804	2084.2842	80176.1581
17	32.1122	2051.0038	78091.8738
18	31.0353	2018.8916	76040.8700
19	30.0454	1987.8562	74021.9784
20	29.1381	1957.8109	72034.1222
21	28.3097	1928.6728	70076.3113
22	27.5563	1900.3631	68147.6386
23	26.8746	1872.8068	66247.2755
24	26.2613	1845.9322	64374.4687
25	25.7134	1819.6709	62528.5365
26	25.2281	1793.9575	60708.8656
27	24.8026	1768.7294	58914.9081
28	24.4344	1743.9268	57146.1787
29	24.1213	1719.4924	55402.2519
30	23.8610	1695.3711	53682.7595
31	23.6514	1671.5101	51987.3885
32	23.4906	1647.8586	50315.8784
33	23.3767	1624.3680	48668.0198
34	23.3080	1600.9913	47043.6517
35	23.2829	1577.6833	45442.6604
36	23.2997	1554.4004	43864.9771
37	23.3569	1531.1008	42310.5767
38	23.4531	1507.7439	40779.4760
39	23.5869	1484.2907	39271.7321
40	23.7569	1460.7038	37787.4414
41	23.9618	1436.9469	36326.7375
42	24.2001	1412.9851	34889.7907
43	24.4705	1388.7850	33476.8056
44	24.7717	1364.3144	32088.0206
45	25.1022	1339.5427	30723.7061
46	25.4604	1314.4406	29384.1634
47	25.8449	1288.9801	28069.7229
48	26.2539	1263.1352	26780.7427
49	26.6857	1236.8813	25517.6075
50	27.1383	1210.1957	24280.7261
51	27.6095	1183.0574	23070.5305
52	28.0972	1155.4478	21887.4731
53	28.5988	1127.3506	20732.0252
54	29.1113	1098.7519	19604.6746
55	29.6319	1069.6405	18505.9227

Age	C_x	M_x	R_x
56	30.1571	1040.0086	17436.2822
57	30.6831	1009.8515	16396.2736
58	31.2057	979.1685	15386.4221
59	31.7204	947.9628	14407.2536
60	32.2223	916.2423	13459.2908
61	32.7057	884.0201	12543.0485
62	33.1646	851.3144	11659.0284
63	33.5925	818.1498	10807.7140
64	33.9824	784.5573	9989.5642
65	34.3265	750.5749	9205.0070
66	34.6167	716.2484	8454.4320
67	34.8444	681.6317	7738.1836
68	35.0005	646.7873	7056.5519
69	35.0755	611.7868	6409.7645
70	35.0597	576.7113	5797.9777
71	34.9434	541.6516	5221.2663
72	34.7168	506.7082	4679.6147
73	34.3706	471.9914	4172.9066
74	33.8959	437.6208	3700.9152
75	33.2850	403.7249	3263.2944
76	32.5312	370.4400	2859.5695
77	31.6300	337.9087	2489.1295
78	30.5786	306.2787	2151.2208
79	29.3771	275.7002	1844.9421
80	28.0289	246.3230	1569.2419
81	26.5407	218.2941	1322.9189
82	24.9234	191.7534	1104.6247
83	23.1918	166.8300	912.8714
84	21.3654	143.6382	746.0413
85	19.4675	122.2729	602.4031
86	17.5254	102.8054	480.1303
87	15.5697	85.2799	377.3249
88	13.6329	69.7102	292.0449
89	11.7485	56.0773	222.3347
90	9.9494	44.3288	166.2574
91	8.2660	34.3795	121.9286
92	6.7248	26.1135	87.5491
93	5.3465	19.3887	61.4356
94	4.1449	14.0421	42.0470
95	3.1256	9.8973	28.0048
96	2.2867	6.7716	18.1075
97	1.6183	4.4850	11.3359
98	1.1043	2.8667	6.8509
99	0.7241	1.7624	3.9842
100	0.4545	1.0384	2.2218
101	0.2719	0.5839	1.1834
102	0.1543	0.3120	0.5995
103	0.0827	0.1577	0.2875
104	0.0416	0.0749	0.1299
105	0.0196	0.0333	0.0549
106	0.0085	0.0138	0.0216
107	0.0034	0.0052	0.0079
108	0.0012	0.0018	0.0026
109	0.0004	0.0006	0.0008
110	0.0001	0.0002	0.0002

NET SINGLE PREMIUMS, $i = 0.06$

Age	$\ddot{a}_x$	$1000\, A_x$	$1000\, ({}^2A_x)$
0	16.80096	49.0025	25.9210
1	17.09819	32.1781	8.8845
2	17.08703	32.8097	8.6512
3	17.07314	33.5957	8.5072
4	17.05670	34.5264	8.4443
5	17.03786	35.5930	8.4547
6	17.01675	36.7875	8.5310
7	16.99351	38.1031	8.6666
8	16.96823	39.5341	8.8553
9	16.94100	41.0757	9.0917
10	16.91187	42.7245	9.3712
11	16.88089	44.4782	9.6902
12	16.84807	46.3359	10.0460
13	16.81340	48.2981	10.4373
14	16.77685	50.3669	10.8638
15	16.73836	52.5459	11.3268
16	16.69782	54.8404	11.8295
17	16.65515	57.2558	12.3749
18	16.61024	59.7977	12.9665
19	16.56299	62.4720	13.6080
20	16.51330	65.2848	14.3034
21	16.46105	68.2423	15.0569
22	16.40614	71.3508	15.8730
23	16.34843	74.6170	16.7566
24	16.28783	78.0476	17.7128
25	16.22419	81.6496	18.7472
26	16.15740	85.4300	19.8657
27	16.08733	89.3962	21.0744
28	16.01385	93.5555	22.3802
29	15.93683	97.9154	23.7900
30	15.85612	102.4835	25.3113
31	15.77161	107.2676	26.9520
32	15.68313	112.2754	28.7206
33	15.59057	117.5148	30.6259
34	15.49378	122.9935	32.6772
35	15.39262	128.7194	34.8843
36	15.28696	134.7002	37.2574
37	15.17666	140.9437	39.8074
38	15.06159	147.4572	42.5455
39	14.94161	154.2484	45.4833
40	14.81661	161.3242	48.6332
41	14.68645	168.6916	52.0077
42	14.55102	176.3572	55.6199
43	14.41022	184.3271	59.4833
44	14.26394	192.6071	63.6117
45	14.11209	201.2024	68.0193
46	13.95459	210.1176	72.7205
47	13.79136	219.3569	77.7299
48	13.62235	228.9234	83.0624
49	13.44752	238.8198	88.7329
50	13.26683	249.0475	94.7561
51	13.08027	259.6073	101.1469
52	12.88785	270.4988	107.9196
53	12.68960	281.7206	115.0885
54	12.48556	293.2700	122.6672
55	12.27581	305.1431	130.6687

Age	$\ddot{a}_x$	$1000\ A_x$	$1000\ (^2A_x)$
56	12.06042	317.3346	139.1053
57	11.83953	329.8381	147.9883
58	11.61327	342.6452	157.3280
59	11.38181	355.7466	167.1332
60	11.14535	369.1310	177.4113
61	10.90412	382.7858	188.1682
62	10.65836	396.6965	199.4077
63	10.40837	410.8471	211.1318
64	10.15444	425.2202	223.3401
65	9.89693	439.7965	236.0299
66	9.63619	454.5553	249.1958
67	9.37262	469.4742	262.8299
68	9.10664	484.5296	276.9212
69	8.83870	499.6963	291.4559
70	8.56925	514.9481	306.4172
71	8.29879	530.2574	321.7850
72	8.02781	545.5957	337.5361
73	7.75683	560.9339	353.6443
74	7.48639	576.2419	370.0803
75	7.21702	591.4895	386.8119
76	6.94925	606.6460	403.8038
77	6.68364	621.6808	421.0184
78	6.42071	636.5634	438.4155
79	6.16101	651.2639	455.9527
80	5.90503	665.7528	473.5861
81	5.65330	680.0019	491.2698
82	5.40629	693.9837	508.9574
83	5.16446	707.6723	526.6012
84	4.92824	721.0431	544.1537
85	4.69803	734.0736	561.5675
86	4.47421	746.7428	578.7956
87	4.25710	759.0320	595.7923
88	4.04700	770.9244	612.5133
89	3.84417	782.4056	628.9163
90	3.64881	793.4636	644.9611
91	3.46110	804.0884	660.6105
92	3.28118	814.2726	675.8298
93	3.10914	824.0111	690.5878
94	2.94502	833.3007	704.8565
95	2.78885	842.1408	718.6115
96	2.64059	850.5325	731.8321
97	2.50020	858.4791	744.5010
98	2.36759	865.9853	756.6047
99	2.24265	873.0577	768.1330
100	2.12522	879.7043	779.0793
101	2.01517	885.9341	789.4400
102	1.91229	891.7573	799.2147
103	1.81639	897.1852	808.4054
104	1.72728	902.2295	817.0170
105	1.64472	906.9025	825.0563
106	1.56850	911.2170	832.5324
107	1.49838	915.1860	839.4558
108	1.43414	918.8224	845.8386
109	1.37553	922.1396	851.6944
110	1.32234	925.1507	857.0377

REFERENCES

Bowers, N. L., Gerber, H. U., Hickman, J. C., Jones, D. A., and Nesbitt, C. J. (1986). *Actuarial Mathematics.* Society of Actuaries, Itasca, Illinois.

Forfar, D. O. and Waters, H. R. (1988). *A Stochastic Approach to Life Contingencies.* Special Note. Institute of Actuaries. Education Service.

Institute of Actuaries (1991). *Formulae and Tables for Actuarial Examinations.* Alden Press, Oxford, England.

Jordan, Jr., C. W. (1975). *Life Contingencies.* Society of Actuaries Textbook. Society of Actuaries, Itasca, Illinois.

McCutcheon, J. J. and Scott, W. F. (1989). *An Introduction to the Mathematics of Finance.* Heinemann Professional Publishing, Oxford, England.

Neill, Alistair (1989). *Life Contingencies.* Heinemann Professional Publication, Oxford, England.

SYMBOL INDEX

SUBJECT INDEX